NON-LINEAR BEHAVIOR OF
FIBER COMPOSITE LAMINATES

by Zvi Hashin, Dinal Bagchi, and B. Walter Rosen

DISCLAIMER NOTICE

THIS DOCUMENT IS BEST QUALITY AVAILABLE. THE COPY FURNISHED TO DTIC CONTAINED A SIGNIFICANT NUMBER OF PAGES WHICH DO NOT REPRODUCE LEGIBLY.

1. Report No.	2. Government Accession No.	3. Recipient's Catalog No.
NASA CR-2313		
4. Title and Subtitle		5. Report Date APRIL 1974
NON-LINEAR BEHAVIOR OF FIBER COMPOSITE LAMINATES		6. Performing Organization Code
7. Author(s)		8. Performing Organization Report No.
ZVI HASHIN, DEBAL BAGCHI, AND B. WALTER ROSEN		TFR/7309
		10. Work Unit No.
9. Performing Organization Name and Address		501-22-03-02
MATERIAL SCIENCES CORPORATION BLUE BELL, PENNSYLVANIA		11. Contract or Grant No. NAS1-11284
		13. Type of Report and Period Covered
12. Sponsoring Agency Name and Address		CONTRACTOR REPORT
NATIONAL AERONAUTICS AND SPACE ADMINISTRATION WASHINGTON, D.C. 20546		14. Sponsoring Agency Code

15. Supplementary Notes

FINAL REPORT

16. Abstract

THE NON-LINEAR BEHAVIOR OF FIBER COMPOSITE LAMINATES WHICH RESULTS FROM LAMINA NON-LINEAR CHARACTERISTICS WAS EXAMINED. THE ANALYSIS USES A RAMBERG-OSGOOD REPRESENTATION OF THE LAMINA TRANSVERSE AND SHEAR STRESS STRAIN CURVES IN CONJUNCTION WITH DEFORMATION THEORY TO DESCRIBE THE RESULTANT LAMINATE NON-LINEAR BEHAVIOR.

A LAMINATE HAVING AN ARBITRARY NUMBER OF ORIENTED LAYERS AND SUBJECTED TO A GENERAL STATE OF MEMBRANE STRESS WAS TREATED. PARAMETRIC RESULTS AND COMPARISON WITH EXPERIMENTAL DATA AND PRIOR THEORETICAL RESULTS ARE PRESENTED.

17. Key Words (Suggested by Author(s))	18. Distribution Statement
	UNCLASSIFIED - UNLIMITED
	CAT. 32

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price*
UNCLASSIFIED	UNCLASSIFIED	130	$4.75

Blank Pages

TABLE OF CONTENTS

LIST OF SYMBOLS

$2a, 2b$	– Laminate dimension;
c	– Fiber volume fraction;
c^*_{ijkl}	– Stiffness matrix;
E_1	– Elastic Young's modulus;
E_A	– Young's modulus in fiber direction;
E_T	– Young's modulus in transverse direction;
F	– Function of stress σ_{ij};
G_1, G	– Elastic shear modulus of matrix;
G_A	– Axial shear modulus;
G^s_A	– Effective secant shear modulus of composite;
$ijkl$	– Subscripts ranging from 1 to 3;
k, m, n	– Ramberg-Osgood parameters for the matrix;
$I_{1,2,3,4,5}$	– Stress invariants;
J_2	– $s_{ij}s_{ij}/2$;
K	– Number of laminae in a laminate;
$L, \underline{L}$	– Matrices, also quadratic function of stresses;
M, N	– Ramberg-Osgood parameters for the composite;
r, θ, z	– Cylindrical coordinate system;
s_{ij}	– Stress deviator;
S_{ij}	– Compliance matrix;
S'_{ij}	– Elastic compliance matrix;
S''_{ij}	– Inelastic compliance matrix;
t	– Laminate thickness;
T_i	– Surface tractions;
$T_{\alpha\beta}$	– Constant edge forces per unit length;
$u_{1,2,3}$	– Displacements in $x_{1,2,3}$ directions, respectively;
$\tilde{u}_i$	– Admissible displacement field;
U_ε	– Strain energy;
U_c	– Complimentary energy;
U_p	– Potential energy;

V	— Volume;
w^ε	— Strain energy density;
w^c	— Complimentary energy density;
$x_{1,2,3}$	— Fixed coordinate directions;
$x^{(k)}_{1,2,3}$	— Local coordinate system for the kth lamina;
α,β	— Subscripts ranging from 1 to 2;
γ	— Shear strain;
ε	— Strain;
$\bar{\varepsilon}_{ij}$	— Average strain tensor;
$\bar{\varepsilon}'_{\alpha\beta}$	— Elastic strains;
$\bar{\varepsilon}''_{\alpha\beta}$	— Inelastic strains;
σ	— Stress;
σ_y	— Nominal composite yield stress;
σ'	— Nominal matrix yield stress;
$\underline{\sigma}^\circ$	— Stress $\sigma^\circ_{\alpha\beta}$ at the edges;
$\bar{\sigma}_{ij}$	— Average stress tensor·
σ°_{ij}	— Applied stress;
τ	— Shear stress;
τ_y	— Nominal composite yield shear stress;
τ_c	— Average shear stress in composite;
τ'_y	— Nominal matrix yield shear stress;
ν_A	— Poisson's ratio; and
θ_k	— Reinforcement angle.

NON-LINEAR BEHAVIOR OF
FIBER COMPOSITE LAMINATES

by Zvi Hashin, Debal Bagchi and B. Walter Rosen
Materials Sciences Corporation

SUMMARY

The non-linear stress-strain behavior of fiber composite laminates has been analyzed to define the relationship between laminate behavior and the non-linear stress-strain characteristics of unidirectional composites. The resulting analysis has been programmed to yield an efficient computerized design and analysis tool.

The approach utilized herein was to adopt a Ramberg-Osgood representation of the non-linear stress-strain behavior and to utilize deformation theory as an adequate representation of the material nonlinearities. The problem was viewed on two levels. First, the relationship between the constituent properties and the stress-strain response of a unidirectional fiber composite material was studied. For this problem, the primary attention was directed toward axial shear behavior, and an expression was established relating the composite average-stress/average-strain curve to the fiber moduli and the matrix non-linear stress-strain curve. Second level of approach is to treat the interelationship between the properties of the unidirectional layers and those of the laminate. For this case, the starting point is a non-linear stress-strain curve for transverse stress and for axial shear and a linear stress-strain relation for stress in the fiber direction. The non-linear lamina stress-strain curves can be modeled by proper selection of the Ramberg-Osgood parameters. In the present study, with this as a starting point, an interaction expression was formulated to account for simultaneous application of axial shear and transverse stress.

A laminate having an arbitrary number of oriented layers and subjected to a general state of membrane stress was treated. Parametric results and comparison with experimental data and prior theoretical results are presented.

1. INTRODUCTION

A basic requirement for the engineer designing with fiber composite materials is a definition of the stiffness and strength of these materials under a variety of loading conditions, including cases for which experimental materials properties data are not available. For this purpose, it is necessary that he have at his disposal reasonably accurate procedures to predict these mechanical properties. Existing analyses can predict the elastic behavior of a laminated composite quite well when the elastic properties of the unidirectional materials from which it is made are known. However, the situation has been much more complicated and much less satisfactory with regard to the inelastic stiffness and strength of a laminate. The present program was undertaken to develop a computerized analysis of the inelastic behavior of fiber composite laminates which could be used as a design tool. The results of this study and comparisons of these results with experimental data are presented in this report.

It is essential to recognize that the utilization of fiber composite materails in structural design involves the incorporation of material design into the structural design process. This is illustrated clearly by the fact that the gross material properties of a fiber composite laminate change when any change is made in the laminate ply orientations. Even when the designer considers a material formed from a particular combination of fiber and matrix materials, there remains a large number of geometric variables associated with the laminate design. Thus, in the preliminary design phase, experimental material properties data will generally be too limited. In the case of elastic properties, sufficient capability to synthesize the necessary properties exists. This procedure generally starts with the definition of the elastic properties of unidirectional fiber composite materials. These can, of course, be determined experimentally. Also, when such data are not available, they can be estimated using a variety of analytical techniques. These

2.

latter are generally referred to as micromechanics analyses. For example, a set of relatively simple relations for predicting the moduli of unidirectional reinforced composites are presented in [1]. Alternate micromechanics approaches are described in [2] to [4]. A review of these methods is presented in [5]. With these properties available, it is assumed that the individual laminae are homogeneous and anisotropic. A laminate analysis is carried out in a straight forward fashion following methods originally developed for such materials as plywood, and more recently extended to the more general cases associated with fiber composite laminates (e.g., [6] to [8]).

However, contemporary fiber composite materials generally consist of elastic brittle fibers such as glass, boron or graphite in relatively soft matrix materials such as epoxy or aluminum. For these matrix materials it is reasonable to anticipate that at a certain loading state the matrix will begin to exhibit inelastic effects. This results in non-linear relations between structural loads and deformations. These inelastic effects can, of course, be expected to have a significant effect upon failure of the laminate. It is quite clear that adequate definition of these failure conditions are essential to achieve structural designs of high reliability.

In the present study, a non-linear laminate analysis has been developed which can provide realistic assessments of the stresses and strains in the various laminae and of the inelastic stiffnesses of the laminate at any stress level. This information can be used for assessment of such effects as structural stability or structural stress distributions. The stress distributions in the laminae and the laminates can also be utilized for the development of more realistic failure criteria.

Inelastic matrix behavior can be classified broadly as either time dependent or time independent. Time dependent behavior is called viscoelastic if linear and creep if non-linear. Polymeric matrices such as epoxy do exhibit such behavior. In

3.

the case of metallic matrix materials, such as aluminum, time
dependent effects are generally negligible unless elevated
temperature conditions are considered. The present study is
concerned with time independent non-linear matrix behavior
which is of significance for both polymeric and metallic matri-
ces. Throughout this paper the expression "inelastic" is used
to describe this time independent mechanical behavior. The
method of approach to these problems is similar to that of the
elastic analysis. Thus, it is necessary to determine, first,
the inelastic properties of the unidirectional fiber composite
materials. This can be done experimentally or by micromechan-
ics methods. Given this information, a method to determine
stresses and strains in an inelastic laminate is then devised.
The problem is complicated by the fact that the inelastic
stress-strain relations are non-linear.

A limited number of pertinent investigations can be found
in the literature. Hill [4] considered, in approximate
fashion, a limited aspect of inelastic behavior of a uniaxially
reinforced material: the case of stress in fiber direction
combined with isotropic transverse stress. Petit and Waddoups
[6] devised an incremental method for laminate analysis in
which it was assumed that in single laminae there is no inter-
action of stress components in different directions as far as
lamina deformation is concerned. This assumption is restric-
tive, and also their incremental laminate analysis scheme is
unduly complicated. Adams [7] used a finite element technique
for numerical analysis of unidirectional materials in the form
of periodic fiber arrays under conditions of plane strain.
Huang [8] gave an approximate analysis for transverse inelas-
tic behavior for a unidirectional material in plane strain,
but it is diffucult to assess the validity of the approxima-
tions introduced.

A detailed analysis of the inelastic laminate problem
has been given by Foye and Baker [9]. Using finite element
methods, they computed the inelastic effective properties of
unidirectional rectangular and square arrays of elastic fibers

4.

in inelastic matrix. These properties were then used in an inelastic laminate analysis. The analysis is based on incremental plasticity theory and is, unfortunately, very complicated and requires a great deal of computer time. The results obtained are, however, of great importance for comparison with results predicted by more simplified theories, such as the one which will be given in the present work.

The body of this report is divided into four major sections. In the first, consideration is given to the behavior of unidirectional fiber composite materials. This requires: a definition of the appropriate form of the inelastic stress-strain relations; some consideration of the relationship between composite properties and constituent properties; and a definition of the appropriate form of the interaction between various stress components. The basic objective in this phase of the report was to define appropriate constitutive relations for the individual lamina which can be used in the non-linear laminate analysis. Further, there is a desire to gain some insight into the influence of the particular constituent properties upon the lamina stress-strain relations. In this phase of the study, it is found useful to characterize the unidirectional material with the aid of Ramberg-Osgood stress-strain relations.

In the next section of the report, the analysis of the inelastic behavior of laminates is described. Here, a procedure for incorporating the non-linear constitutive relations into an analysis which defines the state of stress in the individual laminae under an arbitrary set of external loads, is defined. Analyses are developed for the case of symmetric laminates subjected to membrane loading. The equations which are developed uniquely define the desired laminate internal average stress distribution under a given set of membrane loads. Governing equations, however, are non-linear and require numerical solution procedures. An efficient algorithm has been defined which enables computer solution to be achieved for arbitrary

5.

laminates at minimal cost. The solution is obtained by application of the Newton-Raphson method.

In the final section, the computerized analysis which has been developed is applied to series of problems. The first group presents comparisons with various analytical results from the more complex analyses of Ref. [6] and [9]. The second group of numerical results presents comparisons between theoretical results from the present model and available experimental data. The third group of results provides several parametric studies to gain insight into those factors which contribute significantly to the non-linear behavior of fiber composite laminates. Also, computations have been made to provide a preliminary assessment of combined load effects including comparisons with limited experimental data.

Details of the various analytical developements, as well as descriptions of the computer program, are presented in appendices to the report.

The principal result of the present program is a computer program which provides a simple engineering tool which can be used for the parametric study of the influence of material properties upon laminate performance. This laminate analysis capability can be used by the structural designer to define design allowable stresses and to aid in the selection of fiber composite materials for structural applications. A comparison of the present results with the limited amount of available experimental data shows good agreement. There are, however certain cases in which the agreement is not good, par ticularly as the laminate loading approaches failure. The results of the present analytical method agree well with the results for those problems for which more exact and more complex analytical results exist.

6.

2. NON-LINEAR STRESS-STRAIN RELATIONS OF UNIAXIAL FIBER REINFORCED MATERIALS

2.1 General Form of Stress-Strain Relations

An effective stress-strain relation of a composite material is defined as a relation between average stress $\bar{\sigma}_{ij}$ and average strain $\bar{\epsilon}_{ij}$. Here and in the following latin indices range over 1, 2, and 3. If the composite is elastic the general effective stress-strain relation takes the form

$$\bar{\sigma}_{ij} = C^*_{ijkl}\,\bar{\epsilon}_{kl} \tag{2.1.1}$$

where C^*_{ijkl} are the effective elastic moduli which are material constants and are thus independent of stress or strain. Thus, (2.1.1) is a linear relation between average stress and strain.

If the composite is subject to symmetries the form of (2.1.1) simplifies. For a uniaxial FRM the most important cases of symmetry are transverse isotropy, around fiber direction, and square array (square symmetry). In these cases the stress-strain relations (2.1.1) for transverse isotropy assume the form:

$$\bar{\sigma}_{11} = C^*_{11}\,\bar{\epsilon}_{11} + C^*_{12}\,\bar{\epsilon}_{22} + C^*_{12}\,\bar{\epsilon}_{33}$$

$$\bar{\sigma}_{22} = C^*_{12}\,\bar{\epsilon}_{11} + C^*_{22}\,\bar{\epsilon}_{22} + C^*_{23}\,\bar{\epsilon}_{33}$$

$$\bar{\sigma}_{33} = C^*_{12}\,\bar{\epsilon}_{11} + C^*_{23}\,\bar{\epsilon}_{22} + C^*_{22}\,\bar{\epsilon}_{33}$$

$$\bar{\sigma}_{12} = 2C^*_{44}\,\bar{\epsilon}_{12} \tag{2.1.2}$$

$$\bar{\sigma}_{13} = 2C^*_{55}\,\bar{\epsilon}_{13}$$

$$\bar{\sigma}_{31} = 2C^*_{44}\,\bar{\epsilon}_{31}$$

and

$$C^*_{55} = (C^*_{22} - C^*_{23})/2 \tag{2.1.3}$$

In (2.1.2-3) 1 indicates direction and 2, 3 perpendicular directions transverse to 1.

In the event of inelastic matrix and elastic fibers, the situation is much more complicated since the stress-strain

7.

relation are nonlinearity and history dependent. In no case is stress proportional to strain so that superposition of effects is not valid, and in order to determine current strain it is not sufficient to know current stress but it is necessary to know precisely the variation of stress which preceded its current value. Thus, for a material in a known state of combined shear and uniaxial tension, the state of strain is different if: (a) tension is first applied and then the shear, (b) shear is first applied and then the tension- (c) tension and shear are applied simultaneously. For this reason stress-strain relations must be presented in incremental form. That is, strain increment is related to stress and stress increment. This complicates matters enormously. However, it is known that in the case of proportional loading, that is, all stresses at a point grow simultaneously in a fixed ratio to one another, incremental theory can be integrated into the much simpler total or deformation theory for which current strain is completely determined by current stress.

Deformation theories have a wider range of validity than proportional loading. Comparison of numerous detailed solutions carried out both incrementally and by much simpler deformation theory show surprising agreement in many cases, and Budiansky [10] has shown that deformation theory can also be valid for "neighboring" loading paths.

In the present work, we are concerned with composites which are subjected to some external load. If it is supposed that the various external load components grow proportionally, this does not necessarily imply that the components of stress at a typical internal point also grow proportionally. It is, however, felt that the manner of growth of these internal stress components cannot deviate severely from proportional loading if external loading is proportional. Consequently, deformation type stress-strain relations are assumed for the matrix.

This assumption results in considerable simplification. It will be seen that it yields results which are extremely

8.

close to the ones obtained in [9] on the basis of the much more complicated incremental theory.

It is shown in Appendix A that for elastic fibers and an inelastic matrix described by deformation type theory, the effective stress-strain relations for a transversely isotropic or square symmetric FRM are:

$$\bar{\varepsilon}_{11} = S_{11}\,\bar{\sigma}_{11} + S_{12}\,\bar{\sigma}_{22} + S_{12}\,\bar{\sigma}_{33}$$

$$\bar{\varepsilon}_{22} = S_{12}\,\bar{\sigma}_{11} + S_{22}\,\bar{\sigma}_{22} + S_{23}\,\bar{\sigma}_{33}$$

$$\bar{\varepsilon}_{33} = S_{12}\,\bar{\sigma}_{11} + S_{23}\,\bar{\sigma}_{22} + S_{22}\,\bar{\sigma}_{33} \qquad (2.1.4)$$

$$\bar{\varepsilon}_{12} = 2S_{44}\,\bar{\sigma}_{12}$$

$$\bar{\varepsilon}_{23} = 2S_{55}\,\bar{\sigma}_{23}$$

$$\bar{\varepsilon}_{13} = 2S_{44}\,\bar{\sigma}_{13}$$

and

$$S_{55} = (S_{22} - S_{23})/2 \qquad (2.1.5)$$

The coefficients S_{11}, S_{12}, etc. are the effective inelastic compliances of the material and are <u>functions of the average stresses</u>, or rather of certain invariants of the average stress tensor.

We are here primarily concerned with thin uniaxially reinforced laminae which are in a state of plane stress. Let x_1 denote fiber direction, x_2 direction transverse to fibers in lamina plane, and x_3 direction perpendicular to lamina, Figure 1. Then the plane stress condition is expressed by:

$$\bar{\sigma}_{13} = \bar{\sigma}_{23} = \bar{\sigma}_{33} = 0 \qquad (2.1.6)$$

Equs. (2.1.4) then assume the form:

$$\bar{\varepsilon}_{11} = S_{11}\,\bar{\sigma}_{11} + S_{12}\,\bar{\sigma}_{22} \qquad (2.1.7)$$

$$\bar{\varepsilon}_{22} = S_{12}\,\bar{\sigma}_{11} + S_{22}\,\bar{\sigma}_{22}$$

$$\bar{\varepsilon}_{12} = 2S_{44}\,\bar{\sigma}_{12}$$

9.

Note that $\bar{\varepsilon}_{33}$ does not vanish. It is however of no interest
for present purposes.

The inelastic compliances in (2.1.7) are functions of
the stresses $\bar{\sigma}_{11}$, $\bar{\sigma}_{22}$, $\bar{\sigma}_{12}$.

It is convenient to split the strains in (2.1.7) into
elastic strains $\bar{\varepsilon}'_{\alpha\beta}$, and inelastic strains $\bar{\varepsilon}''_{\alpha\beta}$. Thus:

$$\bar{\varepsilon}_{\alpha\beta} = \bar{\varepsilon}'_{\alpha\beta} + \bar{\varepsilon}''_{\alpha\beta} \tag{2.1.8}$$

where here and in the following greek indices range over 1, 2.
The elastic strains are recovered after unloading of the com-
posite and are related to the stresses by elastic stress-strain
relations. Thus:

$$\bar{\varepsilon}'_{11} = S'_{11} \bar{\sigma}_{11} + S'_{12} \bar{\sigma}_{22}$$
$$\bar{\varepsilon}'_{22} = S'_{12} \bar{\sigma}_{11} + S'_{22} \bar{\sigma}_{22} \tag{2.1.9}$$
$$\bar{\varepsilon}'_{12} = 2S'_{44} \bar{\sigma}_{12}$$

where

$$S'_{11} = \frac{1}{E_A} \qquad\qquad S'_{12} = -\frac{\nu_A}{E_A}$$
$$S'_{22} = \frac{1}{E_T} \qquad\qquad S'_{44} = \frac{1}{4G_A} \tag{2.1.10}$$

Here E_A is the effective Young's modulus in fiber direction,
ν_A - the associated effective Poisson's ration, E_T - the effec-
tive Young's modulus transverse to fibers and G_A - axial effec-
tive shear modulus, related to 1-2 shear.

The inelastic, permanent, strains then have the form:

$$\bar{\varepsilon}''_{11} = S''_{11} \bar{\sigma}_{11} + S''_{12} \bar{\sigma}_{22}$$
$$\bar{\varepsilon}''_{22} = S''_{12} \bar{\sigma}_{11} + S''_{22} \bar{\sigma}_{22} \tag{2.1.11}$$
$$\bar{\varepsilon}''_{12} = 2S''_{44} \bar{\sigma}_{12}$$

where

$$\tag{2.1.12}$$
$$S''_{23} = S''_{23} (\bar{\sigma}_{11}, \bar{\sigma}_{22}, \bar{\sigma}_{12})$$

10.

In order to further simplify the stress-strain relations (2.1.11-.12), some specific features of FRM will be taken into account. In such materials, the fibers are by an order of magnitude stiffer than the matrix (for the case of boron and/or graphite fiber in an epoxy matrix the ratio of fiber to matrix Young's modulus can be in excess of 100). The stiffness ratio becomes larger in the inelastic range since the matrix loses stiffness (i.e., flows) while the fibers retain their stiffness. It is, therefore, clear that the stress $\bar{\sigma}_{11}$ in fiber direction is practically carried by the fibers alone, with insignificant matrix contribution.

On the other hand, the transverse stress $\bar{\sigma}_{22}$ and the shear stress $\bar{\sigma}_{12}$ are primarily carried by the matrix with little fiber contribution.

It follows that inelastic behavior of the FRM is produced primarily by $\bar{\sigma}_{22}$ and $\bar{\sigma}_{12}$ while inelastic behavior for $\bar{\sigma}_{11}$ load can be neglected.

The foregoing comments are summarized into two basic assumptions:

(a) the inelastic strains $\bar{\epsilon}_{22}^{"}$ and $\bar{\epsilon}_{12}^{"}$ are not functions of $\bar{\sigma}_{11}$

(b) the inelastic strain $\bar{\epsilon}_{11}^{"}$ always vanishes.

On the basis of these assumptions, the stress-strain relations (2.1.11-.12) simplify to:

$$
\begin{aligned}
\bar{\epsilon}_{11}^{"} &= 0 \\
\bar{\epsilon}_{22}^{"} &= S_{22}^{"} \, (\bar{\sigma}_{22}, \, \bar{\sigma}_{12}) \; \bar{\sigma}_{22} \\
\bar{\epsilon}_{12}^{"} &= 2S_{44}^{"} \, (\bar{\sigma}_{22}, \, \bar{\sigma}_{12}) \; \bar{\sigma}_{12}
\end{aligned}
\qquad (2.1.13)
$$

2.2 Plane Stress-Strain Relations in Ramberg-Osgood Form

A convenient representation of non-linear one dimensional stress-strain relations has been given by Ramberg and Osgood [11]. For uniaxial stress, for example:

$$\varepsilon = \frac{\sigma}{E_1} [1 - k \, (\frac{\sigma}{\sigma_0})^{m-1}]$$ (2.2.1)

where E_1 represents the elastic Young's modulus, and k, σ', and m are three parameters to be obtained by curve fitting. The parameter σ' is sometimes called nominal yield stress. Equation (2.2.1) represents a family of curves with initial slope E_1, and monotonically decreasing slope with increasing σ. The curves flatten out with increasing m (Fig. 2). Without loss of generality (2.2.1) can be written in the form:

$$\varepsilon = \frac{\sigma}{E_1} [1 + (\frac{\sigma}{\sigma_y})^{m-1}]$$ (2.2.2)

which will be used from now on. Similarly, a stress-strain curve in shear can be represented in the form:

$$\gamma = \frac{\tau}{G_1} [1 + (\frac{\tau}{\tau_y})^{n-1}]$$ (2.2.3)

where G_1 is the elastic shear modulus.

It should be emphasized that (2.2.2-.3) are valid only for one dimensional cases. The question of the generalization to general states of stress and strain has no unique answer. One common used form is isotropic J_2 deformation theory [12].

Next, we consider the case of effective or macroscopic stress-strain relations for the special case of a uniaxially reinforced material in which the matrix in non-linear, with stress-strain relations in Ramberg-Osgood form.

Consider, for example, the case of uniaxial average stress $\bar{\sigma}_{22}$ in direction transverse to fibers, all other average stresses vanish. It then follows from (2.1.7) that:

$$\bar{\varepsilon}_{22} = S_{22} (\bar{\sigma}_{22}) \, \bar{\sigma}_{22}$$ (2.2.4)

12.

Similarly, if the only nonvanishing average stress is $\bar{\sigma}_{12}$, the shear stress-strain relation of the composite is:

$$\bar{\varepsilon}_{12} = 2S_{44}(\bar{\sigma}_{12})\ \bar{\sigma}_{12} \tag{2.2.5}$$

Evidently the inelastic effective compliances S_{22} and S_{44} are functions of the parameters of the inelastic Ramberg-Osgood stress-strain relations of the matrix, of the elastic properties of the fibers and of the internal geometry of the composite. Actual prediction is a very difficult problem. Such problems will be cnsidered in limited fashion in the next paragraph.

Just as matrix stress-strain relations are represented in Ramberg-Osgood form, the same type of curve fitting can also be applied for the effective stress-strain relation of the composite. Thus (2.2.2-.3) are written in the form:

$$\bar{\varepsilon}_{22} = \frac{\bar{\sigma}_{22}}{E_T}\ [1 + (\frac{\bar{\sigma}_{22}}{\sigma_y})^{M-1}] \qquad \text{(a)}$$

$$\bar{\varepsilon}_{12} = \frac{\bar{\sigma}_{12}}{2G_A}\ [1 + (\frac{\bar{\sigma}_{12}}{\tau_y})^{N-1}] \qquad \text{(b)} \qquad (2.2.6)$$

Where E_T is the effective transverse elastic Young's modulus G_A - effective axial elastic shear modulus and σ_y, τ_y, M and N are curve fitting parameters which are in general quite different from the corresponding Ramberg-Osgood matrix parameters.

A question of fundamental and of practical importance is the form of the stress-strain relations for the case of plane stress, taking into account interaction among the various stress components. It should be noted in this repsect that (2.2.6) are special stress-strain relations when $\bar{\sigma}_{22}$ or $\bar{\sigma}_{12}$ act only by themselves.

It is recalled that equations (2.1.13) represent the inelastic parts of the strains for plane stress-strain relations for FRM with stiff fibers. It is shown in Appendix B

13.

that the Ramberg-Osgood form of such plane stress-strain relations is as follows:

$$\bar{\epsilon}_{11}^{"} = 0$$

$$\bar{\epsilon}_{22}^{"} = \frac{\bar{\sigma}_{22}}{E_T} \left[(\frac{\bar{\sigma}_{22}}{\sigma_y})^2 + (\frac{\bar{\sigma}_{12}}{\tau_y})^2 \right]^{\frac{M-1}{2}}$$

$$\bar{\epsilon}_{12}^{"} = \frac{\bar{\sigma}_{12}}{2G_A} \left[(\frac{\bar{\sigma}_{22}}{\sigma_y})^2 + (\frac{\bar{\sigma}_{12}}{\tau_y})^2 \right]^{\frac{N-1}{2}}$$

(2.2.7)

The parameters E_T, G_A, σ_y, τ_y, M, N in (2.2.7) are those of the one dimensional stress-strain relations (2.2.6) which may be regarded as experimentally (or perhaps theoretically) known.

The inelastic parts of the strains are given by (2.1.9-.10), and the total strains are then given by adding equations (2.2.7) and (2.1.9).

Equations (2.2.7) have been compared with computed numerical results given in [9]. Reasonable agreement was obtained. Comparisons for the interaction cases of transverse stress, $\bar{\sigma}_{22}$, versus transverse strain, $\bar{\epsilon}_{22}$, in the presence of axial shear stress, $\bar{\sigma}_{12}$, and axial shear stress, $\bar{\sigma}_{12}$, versus axial shear strain, $\bar{\gamma}_{12}$, are shown in Figures 3 and 4 respectively (in both cases $\bar{\sigma}_{22}/\bar{\sigma}_{12} = 8/3$). It is seen that the agreement is fair for transverse stress-strain relations (Fig. 3) and very good for the shear stress-strain relations (Fig. 4).

Figures 3 and 4 also show the stress-strain relations obtained from Eqs. (2.2.7) for one dimensional transverse tension $\bar{\sigma}_{22}$, and axial shear, $\bar{\sigma}_{12}$, respectively.

14.

2.3 Axial Shear Stress-Strain Relation

This paragraph is concerned with the problem of prediction
of a one dimensional effective axial shear stress-strain re-
lation of a uniaxial FRM in terms of matrix and fiber properties
and the internal geometry of the composite.

The main reason for concentrating on the axial shear prob-
lem is that the inelastic effect is predominant in axial shear
for which significant nonlinearity of the stress-strain response
is obtained (e.g., Figure 4). The effect in fiber direction is
practically non-existent as has indeed been assumed above, and
is relatively small in transverse stress which is shown by the
small curvature of the stress-strain relation in this case
(e.g., Figure 3).

On the basis of all this, it can indeed be assumed as
first approximation that the nonlinearity of the uniaxial FRM
is limited to axial shear alone.

Consider a uniaxially reinforced lamina which is subjected
to pure axial shear, Figure 5, on its surface. The boundary
conditions are:

$$x_3 = \pm t/2 \qquad \sigma_{31} = \sigma_{32} = \sigma_{33} = 0$$

$$x_2 = \pm b \qquad \sigma_{12} = \tau_o \qquad \sigma_{22} = \sigma_{23} = 0 \qquad (2.3.1)$$

$$x_1 = \pm a \qquad \sigma_{12} = \tau_o \qquad \sigma_{11} = \sigma_{13} = 0$$

It may be shown that under such load the only nonvanishing
average stress in the composite is:

$$ \tag{2.3.2} $$

$$\bar{\sigma}_{12} = \tau_o$$

It would seem at first that, given the complexity of the in-
ternal geometry of the composite, the state of stress at any
interior matrix or fiber point is generally three dimensional.
Surprisingly enough, however, this is not so and the only non-
vanishing stress components in the interior of the composites

15.

are the shear stresses σ_{12} and σ_{13}, which are moreover functions of x_2 and x_3 only. Thus the interior state of stress is:

$$\sigma_{12} = \sigma_{12}(x_2, x_3)$$

$$\sigma_{13} = \sigma_{13}(x_2, x_3)$$ (2.3.3)

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{23} = 0$$

The validity of equations (2.3.3) for the case of an elastic composite has been proved in [5]. Their validity for the present much more general inelastic case will be shown elsewhere.

The effective stress-strain relation of the composite in axial shear is defined by:

$$\bar{\epsilon}_{12} = \frac{\sigma_{12}}{2G_A^S} = \frac{\tau}{2G_A^S}$$ (2.3.4)

$$G_A^S = G_A^S(\bar{\sigma}_{12}) = G_A^S(\tau_0)$$

where G_A^S is the <u>effective secant shear modulus</u> of the material. The nonlinearity of the stress-strain relation is expressed by the fact that G_A^S function of the applied stress.

It is seen that in order to determine G_A^S it is necessary to compute the average shear strain $\bar{\epsilon}_{12}$ for given applied shear stress. This is a formidable problem even with the simplification (2.3.3) and we shall content ourselves with a brief outline of its formulation. To simplify matters, the fibers shall be assumed to be ideally rigid relative to the matrix. This is a very accurate assumption for the case of Boron and Graphite Fibers. There is no difficulty to extend the formulation to the case of non-rigid elastic fibers.

In view of (2.3.3) the problem is two dimensional and need only be considered in a typical x_2, x_3 section. In the matrix domain:

16.

$$\frac{\partial \sigma_{12}}{\partial x_2} - \frac{\partial \sigma_{13}}{\partial x_3} = 0 \tag{2.3.5a}$$

$$\varepsilon_{12} = \frac{\sigma_{12}}{2G}\left[1 + \left(\frac{\tau}{\tau_y}\right)^{n-1}\right] \tag{2.3.5b}$$

$$\varepsilon_{13} = \frac{\sigma_{13}}{2G}\left[1 + \left(\frac{\tau}{\tau_y}\right)^{n-1}\right] \tag{2.3.5c}$$

$$\tau = \sqrt{\sigma_{12}^2 + \sigma_{13}^2} \tag{2.3.6}$$

$$\varepsilon_{12} = \frac{1}{2}\frac{\partial u_1}{\partial x_2} \tag{2.3.7a}$$

$$\varepsilon_{13} = \frac{1}{2}\frac{\partial u_1}{\partial x_3} \tag{2.3.7b}$$

$$u_1 = u_1(x_2, x_3) \tag{2.3.8}$$

and, $u_1 = 0$ at fiber/matrix interface.

Here equ. (2.3.5) is the only surviving equilibrium
equation, (2.3.6) are Ramberg-Osgood stress-strain relations
for isotropic J_2 theory (2.3.7) are usual strain-displace-
ment relations in which u_2 and u_3 do not enter since it may
be shown that they are not functions of x_1 and (2.3.8) ex-
presses the ideal rigidity of the fibers.

Equs. (2.3.5-.8) must be solved subject to boundary
condition (2.3.1). If this is done the strain ε_{12} is known
everywhere and can be averaged to obtain G_A^s from (2.3.4).

The problem is exceedingly difficult because of the non-
linearity introduced by the stress-strain relations (2.3.6).
There is very little hope to solve it analytically for any
kind of fiber geometry. It should therefore be handled by
numerical methods for fiber arrangements and fiber shapes
of engineering interest.

Another way to approach an analytically intractable
problem such as the present one is by variational techniques.

17.

In this fashion, approximations or bounds for quantities of interest are obtained by methods which are much simpler than bonafide solution of the problem. Such variational methods have been extensively used for determination of effective elastic moduli of FRM (e.g., [5]).

In the course of the present work, it has been found that variational methods can also be used for inelastic problems such as the present one to obtain bounds on effective secant moduli. The main ingredients of the method are:

(a) Construction of an extremum principle in terms of an energy integral such that the true energy is the minimum of the integral.

(b) Expression of the true energy in terms of effective secant modulus.

(c) Establishment of admissible fields to obtain a value of the energy integral which is larger than the true energy, thus obtaining a bound for G_A^S.

The work involves complicated developments and derivations which are given in Appendix C. Here only the end result for a lower bound on G_A^S will be given for a special geometry of FRM which is known as composite cylinder assemblage. This geometry has been described in detail in [1, 5] and consists of an assemblage of composite cylinders of variable sizes which are joined together so as to fill the whole volume of the composite. In order to fill the whole volume, composite cylinders vary from finite to infinitesimal size. This geometry has been used to advantage for elastic FRM to obtain simple expressions for effective elastic moduli which are well verified by experiment [1, 5]. In the present case only a lower bound on G_A^S has been obtained for the case in which the exponent n in matrix stress-strain relations is n=3.

It has been found that with this exponent and proper choice of τ_y, epoxy shear stress-strain relations can be well described. The result for the lower bound is:

18.

$$G_A^S \geq G_{A(-)}^S = \frac{G \frac{1+c}{1-c}}{1 + (\frac{\tau_o}{\tau_y'})^2 \frac{3+13c+c^2+c^3}{3(1-c)^3}}$$ (2.3.9)

where

   c - volume fraction of fibers

   G - elastic (initial) matrix shear modulus

   τ_y' - Ramberg-Osgood matrix stress parameter, and

   τ_o - applied shear stress.

It follows from (2.3.4) that:

$$\bar{\varepsilon}_{12} \leq \frac{\tau_o}{2G_{A(-)}^S}$$ (2.3.10)

In other words, with the lower bound on G_A^S an upper bound on $\bar{\varepsilon}_{12}$ variation with τ_o is obtained.

If (2.3.10) is explicitly written in terms of (2.3.9) it assumes the form:

$$\bar{\varepsilon}_{12} \leq \frac{\tau_o}{2G\frac{1+c}{1-c}} [1 + (\frac{\tau_o}{\tau_y'})^2 \frac{3+13c+c^2+c^3}{3(1+c)^3}]$$ (2.3.11)

Recalling that for the composite cylinder assemblage with rigid fibers the axial elastic shear modulus G_A is given in [1, 5] as:

$$G_A = G \frac{1+c}{1-c}$$ (2.3.12)

19.

and comparing (2.3.11) with (2.2.6) with choice of exponent (which is the same as matrix exponent), it is seen that:

$$\bar{\gamma} = \bar{\gamma}^{\ast\ast}\,\frac{3(1+c)^3}{3+13c+c^2+c^3}$$

(2.3.13)

The prediction of (2.3.11) has been compared with numerical results obtained in [9]. Figure 6 shows the variation of the right side of (2.3.11) in comparison with the results obtained in [9] for a fiber volume fraction, c=0.5. Since results of [9] were for boron fibers in epoxy matrix, the rigid fiber approximation is accurately valid. It is seen that the results are reasonably close. It should be noted that the geometry of [9] is a rectangular fiber array which is quite different from the composite cylinder assemblage geometry.

The results defined by (2.3.12) and (2.3.13) used in equation (2.2.6) yield the result plotted in non-dimensional form in Fig. 7. The shear strains are normalized with respect to the matrix elastic strain, γ_{ye}, at the yield stress, τ_y:

$$\gamma_{ye} = \frac{\tau_y}{G}$$

(2.3.14)

It is natural to also consider the establishment of an upper bound on G_A^S. Unfortunately, however, this is a matter of formidable difficulty for the reason that inversion of (2.3.6) to express stresses in terms of strains leads very complicated expressions. Further discussion of this difficulty is given in Appendix C.

20.

3. ANALYSIS OF NON-LINEAR LAMINATES

3.1 Formulation

The general problem to be investigated in the present chapter is as follows: given the inelastic stress-strain relations of uniaxially reinforced laminae determined theoretically or experimentally, and a laminate composed of such laminae and loaded on its edges by uniformly distributed loads in the plate of the laminate:

(a) What are the stresses in the various laminae?

(b) What is the macroscopic strain response of the laminate to the loads?

This problem has been extensively investigated for elastic laminates, and the results obtained will serve as important guidelines for the present much more complicated problem. It is therefore very helpful to first briefly review the theory of elastic laminates.

Let the laminate be referred to a fixed system of coordinates x_1, x_2, x_3 as shown in Figure 8. This will henceforth be referred to as the laminate coordinate system.

Any lamina, kth say, in the laminate will be referred to its material system of coordinates $x_1^{(k)}$, $x_2^{(k)}$, x_3 where $x_1^{(k)}$ is in fiber direction, $x_2^{(k)}$ perpendicular to fiber direction and x_3 is the same as the laminate x_3, Figure 8. The reinforcement angle θ_k is defined by:

$$\theta = \sphericalangle (x_1, x_1^{(k)}) = \sphericalangle (x_2, x_2^{(k)}) \tag{3.1.1}$$

Let it be assumed that the laminae are in states of plane stress. It will be later explained under what conditions this is true. Then the stress-strain relations of a single lamina referred to its material coordinate system are written in the forms:

$$\varepsilon_{\alpha\beta}^{(k)} = S_{\alpha\beta\gamma\delta}^{(k)} \sigma_{\gamma\delta}^{(k)} \qquad \text{(a)}$$

$$\underline{\varepsilon}^{(k)} = \underline{S}^{(k)} \underline{\sigma}^{(k)} \qquad \text{(b)} \tag{3.1.2}$$

21.

where (3.1.2a) is in tensor notation with range of subscripts 1, 2 and (3.1.2b) is in matrix notation. It should be noted that (3.1.2) represent the stress-strain relations (2.1.9 - .10), i.e.,

$$\varepsilon_{11}^{(k)} = \frac{\sigma_{11}^{(k)}}{E_A^{(k)}} - \frac{\nu_A^{(k)}}{E_A^{(k)}} \sigma_{22}^{(k)}$$

$$\varepsilon_{22}^{(k)} = - \frac{\nu_A^{(k)}}{E_A^{(k)}} \sigma_{11}^{(k)} + \frac{\sigma_{22}^{(k)}}{E_T^{(k)}} \qquad (3.1.3)$$

$$\varepsilon_{12}^{(k)} = \frac{\sigma_{12}^{(k)}}{2G_A^{(k)}}$$

Let a laminate of rectangular form, Figure 8, be loaded by a uniform edge stress:

$$\sigma_{11}(\pm a, x_2) = \sigma_{11}^{o}$$

$$\sigma_{12}(\pm a, x_2) = \sigma_{12}^{o}$$

$$\sigma_{12}(x_1, \pm b) = \sigma_{12}^{o} \qquad (3.1.4)$$

$$\sigma_{22}(x_1, \pm b) = \sigma_{22}^{o}$$

The elasticity solution of the laminate must satisfy the following requirements:

(a) Equilibrium of stresses,

(b) Traction continuity at laminae interfaces,

(c) Boundary conditions (3.1.4), and

(d) Displacement continuity at laminae interfaces.

It is assumed that the stresses in any lamina are constant, but different in the different laminae. The condition (a) is satisfied within any lamina. Since the assumed lamina stresses are plane there are no traction components on laminae interfaces. Therefore (b) is satisfied.

The boundary conditions (3.1.4) cannot be strictly satisfied in each lamina but only in an average sense. To do this lamina stresses $\sigma_{ij}^{(k)}$ referred to lamina material coordinates

22.

are transformed to laminate axes. The stresses in the kth lamina referred to laminate axes are denoted $^{(k)}\sigma_{\alpha\beta}$. The transformation is given by:

$$^{(k)}\sigma_{11} = \sigma_{11}^{(k)} \cos^2\theta_k + \sigma_{22}^{(k)} \sin^2\theta_k - 2\sigma_{12}^{(k)} \cos\theta_k \sin\theta_k$$

$$^{(k)}\sigma_{22} = \sigma_{22}^{(k)} \sin^2\theta_k + \sigma_{22}^{(k)} \cos^2\theta_k + 2\sigma_{12}^{(k)} \cos\theta_k \sin\theta_k \qquad (3.1.5)$$

$$^{(k)}\sigma_{12} = (\sigma_{11}^{(k)} - \sigma_{22}^{(k)}) \sin\theta_k \cos\theta_k + \sigma_{12}^{(k)} (\cos^2\theta_k - \sin^2\theta_k)$$

or in matrix notation:

$$^{(k)}\underline{\sigma} = \underline{\theta}^{(k)} \underline{\sigma}^{(k)} \qquad (3.1.6)$$

Let the edges of the laminate be loaded by constant forces per unit length T_{11}, T_{22}, T_{12} and define the stresses (3.1.4) as edge averages over the laminate thickness h:

$$\sigma_{11}^{o} = T_{11}/h$$
$$\sigma_{22}^{o} = T_{22}/h \qquad (3.1.7)$$
$$\sigma_{12}^{o} = T_{12}/h$$

Equilibrium requires that:

$$\sum_{k=1}^{K} \sigma_{11}^{(k)} = \sigma_{11}^{o}$$

$$\sum_{k=1}^{K} \sigma_{22}^{(k)} = \sigma_{22}^{o} \qquad (3.1.8)$$

$$\sum_{k=1}^{K} \sigma_{12}^{(k)} = \sigma_{12}^{o}$$

where K is the number of laminae. Written in terms of stresses $\sigma_{\alpha\beta}^{(k)}$ using (3.1.6), we have:

$$\sum_{k=1}^{K} \underline{\theta}^{(k)} \underline{\sigma}^{(k)} = \underline{\sigma}^{o} \qquad (3.1.9)$$

where $\underline{\sigma}^{o}$ denotes the stresses $\sigma_{\alpha\beta}^{o}$ at the edges.

23.

Replacement of the boundary conditions (3.1.4) by (3.1.6) is an approximation of Saint Venant type. Thus, there must be expected edge perturbations (among them interlaminar shear) on the stresses predicted by laminate theory.

Equations (3.1.8) are three equations for the 3K stresses $\sigma_{\alpha\beta}^{(1)}$, $\sigma_{\alpha\beta}^{(2)}\ldots\sigma_{\alpha3}^{(K)}$ in the laminae. There are needed an additional 3(K-1) equations which are provided by displacement continuity at lamina interfaces, requirement (d).

Since the stresses in each laminae are by hypothesis uniform, so are the strains. Therefore, displacement continuity is ensured if the lamine strains in adjacent laminae, referred to laminae coordinate system are the same. Thus:

$$^{(k)}\varepsilon_{11} = {}^{(k+1)}\varepsilon_{11}$$
$$^{(k)}\varepsilon_{22} = {}^{(k+1)}\varepsilon_{22} \qquad k=1,2\ldots\ldots k$$
$$^{(k)}\varepsilon_{12} = {}^{(k+1)}\varepsilon_{12}$$

(3.1.10)

Equations (3.1.10) are the additional required 3(K-1) equations. They will be written in terms of laminae stresses $\sigma_{\alpha\beta}^{(k)}$ referred to laminae material axes. To do this it is noted that:

$$^{(k)}\underline{\varepsilon} = \underline{\theta}^{(k)}\,\underline{\varepsilon}^{(k)}$$

which is just a transformation of (3.1.6). From (3.1.2b):

$$^{(k)}\underline{\varepsilon} = \underline{\theta}^{(k)}\,\underline{S}^{(k)}\,\underline{c}^{(k)} \tag{3.1.11}$$

and inserting the last result in (3.1.10):

$$\underline{\theta}^{(k)}\,\underline{S}^{(k)}\,\underline{c}^{(k)} = \underline{\theta}^{(k+1)}\,\underline{S}^{(k+1)}\,\underline{c}^{(k+1)} \qquad k=1,2\ldots\ldots k \tag{3.1.12}$$

Equations (3.1.9) and (3.1.12) are 3K linear equations for the 3K stresses in an elastic laminate, with K layers.

It should be carefully noted that the analysis given above is based on plane stress conditions in individual laminae. This

24.

is a valid assumption if:

    (a)   The loads on the laminate are statically equiv-
           ient to in-plane forces (membrane forces) and pro-
           duce neither bending nor twisting moments, and

    (b)   The laminate has a certain stacking sequence of
           laminae which defines a so called balanced or
           <u>symmetric laminate</u>.

This stacking sequence is an arrangement in which the
laminate has a middle plane of geometrical and of material
symmetry. The laminae are arranged in paris with respect to
the plane of symmetry. The laminae of such pair have equal
thicknesses, same distances from middle plane, and are of
the same material with same angles of reinforcement.

In a non-symmetric laminate application of membrane
forces will in general produce bending and twisting of lam-
inae and thus a plane state of stress will not be realized.
The symmetric laminate is, however, sufficiently versatile
to cover most cases of practical interest.

Let it now be assumed that the laminate is inelastic but
still fulfills the conditions of symmetry and pure membrane
loading. In this case the only equations which necessarily
change in the preceding development are the stress-strain re-
lations of the laminae, (3.1.2), which must be replaced by in-
elastic laminae stress-strain relations are given by (2.1.7)
where the compliances are now functions of the stresses. These
compliances now replace the elastic compliances in (3.1.2)
which thus become non-linear.

It is convenient for later purposes to rewrite (3.1.2) in
the inelastic case in different form. To do this the strains
$\varepsilon_{\alpha\ell}^{(k)}$ are first split into elastic strains (2.1.9) and inelas-
tic strains (2.1.11). Preceeding to (3.1.12) this equation
assumes the form:

$$\underline{\theta}^{(k+1)}\underline{S}^{1(k+1)}\underline{\sigma}^{(k+1)} - \underline{\theta}^{(k)}\underline{S}^{1(k)}\underline{c}^{(k)} =$$
$$- \underline{\theta}^{(k+1)}\underline{S}^{11(k+1)}\underline{\sigma}^{(k+1)} + \underline{\theta}^{(k)}\underline{S}^{11(k)}\underline{c}^{(k)} \qquad (3.1.13)$$

$$k = 1, 2 \ldots\ldots k$$

where

$\underline{S}^{1(k)}$ - elastic compliance matrix of kth layer

$\underline{S}^{11(k)}$ - inelastic part of compliance matrix of kth layer

$$\underline{S}^{(k)} = \underline{S}^{1(k)} + \underline{S}^{11(k)} \tag{3.1.14}$$

Equations (3.1.13) are now written out in component form with notation (2.1.10), (2.1.12) for compliances:

$$\sigma_{11}^{(k+1)} (S_{11}^{1\ (k+1)} \cos^2\theta_{k+1} + S_{12}^{1(k+1)} \sin^2\theta_{k+1})$$

$$+ \sigma_{22}^{(k+1)} (S_{12}^{1(k+1)} \cos^2\theta_{k+1} + S_{22}^{1(k+1)} \sin^2\theta_{k+1})$$

$$- 4\sigma_{12}^{(k+1)} S_{44}^{1(k+1)} \cos\theta_{k+1}\sin\theta_{k+1} - \sigma_{11}^{(k)} (S_{11}^{1(k)}\cos^2\theta_k + S_{12}^{1(k)} \sin^2\theta_k)$$

$$+ \sigma_{22}^{(k)} (S_{12}^{1(k)}\cos^2\theta_k + S_{22}^{1} \sin^2\theta_k) = 4\sigma_{12}^{(k)} S_{44}^{1(k)} \cos\theta_k \sin\theta_k$$

$$= - \sigma_{11}^{(k+1)} (S_{11}^{11(k+1)}\cos^2\theta_{k+1} + S_{12}^{11(k+1)}\sin^2\theta_{k+1})$$

$$+ \sigma_{22}^{(k+1)} (S_{12}^{11(k+1)}\cos^2\theta_{k+1} + S_{22}^{11(k+1)}\sin^2\theta_{k+1} - 4\sigma_{12}^{(k+1)}S_{44}^{11(k+1)}\cos\theta_{k+1}\sin\theta_{k+1}$$

$$+ \sigma_{11}^{(k)} (S_{11}^{11(k)}\cos^2\theta_k + S_{12}^{11(k)}\sin^2\theta_k) + \sigma_{22}^{(k)} (S_{12}^{11(k)}\cos^2\theta_k + S_{22}^{11}\sin^2\theta_k)$$

$$- 4\sigma_{12}^{(k)} S_{44}^{11(k)}\cos\theta_k\sin\theta_k \tag{3.1.15}$$

26.

$$\sigma_{11}^{(k+1)} (S_{11}^{1(k+1)} \sin^2\theta_{x+1} + S_{12}^{1(k+1)} \cos^2\theta_{x-1}$$

$$+\sigma_{22}^{(k+1)} (S_{12}^{1(k+1)} \sin^2\theta_{x+1} + S_{22}^{1(k+1)} \cos^2\theta_{x-1})$$

$$+4\sigma_{12}^{(k+1)} S_{44}^{1(k+1)} \cos\theta_{x-1} \sin\theta_{x+1}$$

$$\sigma_{11}^{(k)} (S_{11}^{1(k)} \sin^2\theta_k - S_{12}^{1(k)} \cos^2\theta_x$$

$$+\sigma_{22}^{(k)} \quad {}_{12} \quad \sin^2\theta_k + S_{22}^{1(k)} \cos^2\theta_k$$

$$+4\sigma_{12}^{(k)} S_{12}^{1(k)} S_{44}^{1(k)} \cos\theta_x \sin\theta_x$$

$$= - \sigma_{11}^{(k+1)} (S_{11}^{11(k+1)} \sin^2\theta_{x+1} + S_{12}^{11(k+1)} \cos^2\theta_{x+1})$$

$$+\sigma_{22}^{(k+1} (S_{12}^{11(k+1)} \sin^2\theta_{k+1} + S_{22}^{11(k+1)} \cos^2\theta_{k+1}$$

$$+4\sigma_{12}^{(k+1)} S_{44}^{11(k+1)} \cos_{k+1} \sin\theta_{k+1} + \sigma_{11}^{(k)} (S_{11}^{11(k)} \sin\theta_k + S_{12}^{11(k)} \cos^2\theta_x)$$

$$-\sigma_{22}^{(k)} (S_{12}^{11(k)} \sin^2\theta_k + S_{22}^{11(k)} \cos^2\theta_x + 4\sigma_{12}^{(k)} S_{44}^{11(k+1)} \cos_{x+1} \sin\theta_{k+1}$$

(3.1.16)

$$\sigma_{11}^{(k+1)} (S_{11}^{1(k-1)} - S_{22}^{1(x+1)} \sin\theta_{k+1} \cos\theta_{x+1}$$

$$+\sigma_{22}^{(k+1)} (S_{12}^{1(k+i)} - S_{22}^{1(k+1)}) \sin\theta_{k+1} \cos\theta_{x+1}$$

$$+2\sigma_{12}^{(k+1)} S_{44}^{1(k+1)} (\cos^2\theta_{x+1} - \sin^2\theta_{x+1})$$

$$- \sigma_{11}^{(k)} (S_{11}^{1(k)} - S_{12}^{1(k)}) \sin\theta_k \cos\theta_x$$

$$- \sigma_{22}^{(k)} (S_{12}^{11(k)} - S_{22}^{1(k)}) \sin\theta_k \cos\theta_k + 2\sigma_{12}^{(k)} S_{44}^{1(k)} (\cos^2\theta_k - \sin^2\theta_k)$$

$$= - \sigma_{11}^{(k+1)} (S_{11}^{11(k+1)} - S_{12}^{11(k+1)}) \sin\theta_{x+1} \cos\theta_{k+1}$$

$$+\sigma_{22}^{(k+1)} (S_{12}^{11(k+1)} - S_{22}^{11(k+1)}) \sin\theta_k \cos\theta_x$$

$$+2\sigma_{12}^{(k+1)} S_{44}^{1)(k+1)} (\cos^2\theta_{k+1} - \sin^2\theta_{x+1})$$

$$+ \sigma_{11}^{(k)} (S_{11}^{11(k)} - S_{12}^{11(k)}) \sin\theta_k \cos\theta_k$$

$$+\sigma_{22}^{(k)} (S_{12}^{11(k)} - S_{22}^{11(k)}) \sin\theta_k \cos\theta_x$$

$$+2\sigma_{12}^{(k)} S_{44}^{11(k)} (\cos^2\theta_x - \sin^2\theta_k)$$

(3.1.17)

$$k = 1,2 \ldots .. k-1$$

27.

To these must be adjoined equations (3.1.9) which are written here in components:

$$\sum_{k=1}^{K}(\sigma_{11}^{(k)}\cos^2\theta_k+\sigma_{22}^{(k)}\sin^2\theta_k-2\sigma_{12}^{(k)}\cos\theta_k\,\sin\theta_k)t_k = \sigma_{11}^{o}\,h \qquad (a)$$

$$\sum_{k=1}^{K}(\sigma_{11}^{(k)}\sin^2\theta_k+\sigma_{22}^{(k)}\cos^2\theta_k+2\sigma_{12}^{(k)}\cos\theta_k\,\sin\theta_k)\,t_k = \sigma_{22}^{o}\,h \qquad (b)$$

$$\sum_{k=1}^{K}(\sigma_{11}^{(k)}-\sigma_{22}^{(k)})\cos\theta_k\sin\theta_k+\sigma_{12}^{(k)}(\cos^2\theta_k-\sin^2\theta_k)t_k = \sigma_{12}^{o}\,h \qquad (c)$$

$$(3.1.18)$$

We now consider special cases of interest. In the first case the inelastic laminae strains have the form (2.1.13). Then the right side of (3.1.15-.17) simplifies by setting:

$$S_{11}^{11(k)} = S_{11}^{11(k+1)} = S_{12}^{11(k)} = S_{12}^{11(k+1)} = 0$$

$$S_{22}^{11(k)} = S_{22}^{11(k)}\,(\sigma_{22}^{(k)},\;\sigma_{12}^{(k)})$$

$$S_{22}^{11(k+1)} = S_{22}^{11(k+1)}\,(\sigma_{22}^{(k+1)},\;\sigma_{12}^{(k+1)})$$

$$S_{44}^{11(k)} = S_{44}^{11(k)}\,(\sigma_{22}^{(k)},\;\sigma_{12}^{(k)})$$

$$S_{44}^{11(k+1)} = S_{44}^{11(k+1)}\,(\sigma_{22}^{(k+1)},\;\sigma_{12}^{(k+1)})$$

$$(3.1.19)$$

Once the stresses in the laminae have been obtained the strains in the laminae, referred to laminate axes, are determined from (3.1.11). Since the strains in all laminae are the same when referred to the laminate coordinate system, these

28.

are also the average laminate strains and thus determine the inelastic response of the laminate.

In the simplest case the lamina material is assumed to be inelastic in shear only. In that event we have in addition to (3.1.19):

$$S_{22}^{''}(k) - S_{22}^{''}(k+1) = 0 \qquad (3.1.20)$$

and for Ramberg-Osgood presentation of inelastic part of shear compliance:

$$2S_{44}^{11(k)} = \frac{1}{2G_A^{(k)}} \left(\frac{c_{12}^{(k)}}{\tau_y^{(k)}}\right)^{N_k - 1}$$

$$2S_{44}^{11(k+1)} = \frac{1}{2G_A^{(k+1)}} \left(\frac{\sigma_{12}^{(k+1)}}{\tau_y^{(k+1)}}\right)^{N_{k+1} - 1} \qquad (3.1.21)$$

In Ramberg-Osgood representation (2.2.1) the inelastic parts of the compliances assume forms such as:

$$S_{22}^{11(k)} = \frac{1}{E_T^{(k)}} \left[\left(\frac{\sigma_{22}^{(k)}}{\sigma_y^{(k)}}\right)^2 + \left(\frac{\sigma_{12}^{(k)}}{\tau_y^{(k)}}\right)^2 \right]^{1/2(N_k - 1)}$$

$$2S_{44}^{11(k)} = \frac{1}{2G_A^{(k)}} \left[\left(\frac{\sigma_{22}^{(k)}}{\sigma_y^{(k)}}\right)^2 + \left(\frac{\sigma_{12}^{(k)}}{\tau_y^{(k)}}\right)^2 \right]^{1/2(N_k - 1)}$$

$$(3.1.22)$$

$$S_{22}^{11(k+1)} = \frac{1}{E_T^{(k+1)}} \left[\left(\frac{\sigma_{22}^{(k+1)}}{\sigma_y^{(k+1)}}\right)^2 \left(\frac{\sigma_{12}^{(k+1)}}{\tau_y^{(k+1)}}\right)^2 \right]^{1/2(N_{k+1} - 1)}$$

$$2S_{44}^{11(k+1)} = \frac{1}{2G_A^{(k+1)}} \left[\left(\frac{\sigma_{22}^{(k+1)}}{\sigma_y^{(k+1)}}\right)^2 + \left(\frac{\sigma_{12}^{(k+1)}}{\tau_y^{(k+1)}}\right)^2 \right]^{1/2(N_k - 1)}$$

3.2 Method of Solution

The equations which define the laminae stresses are (3.1.9) and (3.1.13) in condensed form, or equivalently, (3.1.15- 3.1.17), (3.1.18) in full form. To explain the solution method it is simpler to write in terms of the condensed form.

Define the matrices:

$$\underline{L}^{1(k+1)} = \underline{\theta}^{(k+1)} \underline{S}^{1(k+1)}$$

$$\underline{L}^{11(k+1)} = \underline{\theta}^{(k+1)} \underline{S}^{11(k+1)} \qquad\qquad (3.2.1)$$

$$\underline{L}^{1(k)} = \underline{\theta}^{(k)} \underline{S}^{1(k)}$$

$$\underline{L}^{11(k)} = \underline{\theta}^{(k)} \underline{S}^{11(k)}$$

Then equs. (3.1.13) assume the form:

$$\underline{L}^{1(k+1)} \underline{\sigma}^{(k+1)} - \underline{L}^{1(k)} \underline{\sigma}^{(k)} = -\underline{L}^{11(k+1)} \underline{\sigma}^{(k+1)} + \underline{L}^{11(k)} \underline{\sigma}^{(k)} \quad (3.2.2)$$

to which are adjoined equs. (3.1.9) which are here rewritten:

$$\sum_{k=1}^{K} \underline{\theta}^{(k)} \underline{\sigma}^{(k)} = \underline{\sigma}^{\circ} \qquad\qquad (3.2.3)$$

The equations may be solved numerically by an iteration method which proceeds as follows: Consider equs. (3.2.2-3) with the right side of (3.2.2) zero. This defines a set of stresses $\underline{\sigma}_o^{(k)}$ given by:

$$\underline{L}^{1(k+1)} \underline{\sigma}_o^{(k+1)} - \underline{L}^{1(k)} \underline{\sigma}_o^{(k)} = 0 \qquad\qquad (a)$$

$$k = 1,2 \ldots .k=1 \qquad (3.2.4)$$

$$\sum_{k=1}^{K} \underline{\theta}^{(k)} \underline{\sigma}_o^{(k)} = \underline{\sigma}^{\circ} \qquad\qquad (b)$$

Since (3.2.4a) contains only elastic compliances $S^{,(k)}$ it is seen that the equations are linear and define the stresses in an _elastic_ laminate. Now insert the stresses $\underline{\sigma}_o^{(k)}$ into the right side of (3.2.2) and define the stresses $\underline{\sigma}_1^{(k)}$ by:

30.

$$\underline{\underline{L}}^{1\,(k+1)}\underline{z}_1^{(k+1)} - \underline{\underline{L}}^{1\,(k)}\underline{c}_1^{(k)} = -\underline{\underline{L}}^{11\,(k+1)}[\underline{c}_o^{(k+1)}]\underline{z}_o^{(k+1)} \quad \text{(a)}$$

$$+\underline{\underline{L}}^{11\,(k)}[\underline{c}_o^{(k)}]\,\underline{z}_o^{(k)} \tag{3.2.5}$$

$$\sum_{k=1}^{K} \underline{\underline{\theta}}^{(k)}\underline{z}_1^{(k)} = \underline{z}^o \quad \text{(b)}$$

Equs. (3.2.5) defines (hopefully) a new approximation $\underline{z}_1^{(k)}$ which is the solution of a set of linear equations. The stresses in square brackets in the right side of (3.2.5) are to emphasize the stress dependence of the non-linear parts of the compliances.

The procedure just initiated can be repeated indefinitely. In general:

$$\underline{\underline{L}}^{1\,(k+1)}\underline{c}_{\ell+1}^{(k+1)} - \underline{\underline{L}}^{1\,(k)}\underline{\sigma}_{\ell+1}^{(k)} = -\underline{\underline{L}}^{11\,(k+1)}[\underline{c}_\ell^{(k+1)}]\,\underline{\sigma}_\lambda^{(k+1)}$$

$$+\underline{\underline{L}}^{11\,(k)}[\underline{\sigma}_\lambda^{(k)}]\,\underline{c}_\ell^{(k)} \tag{3.2.6}$$

$$\sum_{k=1}^{K} \underline{\underline{\theta}}^{(k)}\underline{\sigma}_{\ell+1}^{(k)} = \underline{\sigma}^o$$

This iteration procedure is quite easy to carry out with aid of a computer. It replaces the solution of a set of non-linear equations by solution of a sequence of linear equations, provided of course, that convergence is obtained.

It should be noted that the first iteration step does not necessarily have to start with equs. (3.2.4a), i.e., with zero right side of (3.2.2). Any stresses $\underline{c}_i^{(k)}$ which fulfill (3.2.4b) can be used to start the iteration with (3.2.5) and continuing with the general iteration relation (3.2.6).

It is desired to obtain a laminate solution for only one load system $\underline{c}^o$ then it would seem most logical to start with (3.2.4). But suppose there is a sequence of loadings $\Delta\underline{c}^o$, $2\Delta\underline{\sigma}^o\ldots n\Delta\underline{c}_o$. Suppose that a solution for $(n-1)\,\Delta\underline{c}^o$ has been obtained and that a solution for $n\Delta\underline{\sigma}^o$ is desired. One possibility is to multiply all stresses due to the load $(n-1)\,\Delta\underline{\sigma}^o$ by the factor $n/(n-1)$. The stresses thus obtained certainly

also satisfy (3.2.6b) because of the linearity of these equations. They will generally be reasonable starting values $\underline{z}_o^{(k)}$ for the iteration.

This method of iteration to obtain a solution was found to work well for many sample problems; however, there were cases in which the solution did not converge. Attempts to modify the recurrence relations to overcome this problem met with only partial success. Thus, an alternate procedure for solution was defined. The solution was obtained by application of the Newton-Raphson method.

The set of 3K nonlinear equations represented by equs. (3.2.2-.3) may be presented in the form:

$$F_n \ (c_{ij}^{\ k}) = 0 \qquad n = 1,2 \ ...3K \qquad (3.2.7)$$

The function F_i is expanded in a Taylor series about an arbitrary set of initial stresses which may be taken as the solutions of the elasticity problem. Considering only two terms of the series, it is found that

$$F_i = F_i^c + \frac{\partial F_i^o}{\partial \tau_{mn}^{\ k}} \ \Delta\tau_{mn}^{\ k} = 0 \qquad (3.2.8)$$

or

$$c_{ij}^{\ k} = {}^o c_{ij}^{\ k} - [\frac{\partial F_n^o}{\partial \sigma_{ij}^{\ k}}]^{-1} \ F_n^{\ o} \qquad (3.2.9)$$

where σ_{ij}^k is the corrected solution obtained from the assumed solution ${}^o c_{ij}^k$. Using σ_{ij}^k as the initial guess, the process is repeated until the result is obtained within a desired accuracy. A recurrence form of equation (3.2.9) to obtain the stresses at t+1 cycle from t cycle can be constructed as follows:

32.

$$(\sigma_{ij}^{k})_{t+1} = (\sigma_{ij}^{k})_{t} - [\frac{\partial F_{n}}{\partial \sigma_{ij}^{k}}]^{-1} (F_{n})_{t}$$

After the stresses σ_{ij}^{k} are obtained for all layers of the laminate, strains for any layer k in terms of laminae axes can be computed using equs. (3.1.3). Strains in terms of the laminate axes can be obtained using the strain transformation law.

This analysis has been developed into an efficient computer program. A description of the program including a listing, is presented in Appendix E.

3.3 Numerical Results

The computer program which has been developed under the present study has been utilized in the analysis of a variety of different composite laminates. The initial studies using the computerized analysis were directed at presenting a comparison between the results of the present analysis and those of previous analyses, notably that of Ref. 9. (The present results were also compared to available experimental data, primarily those of Ref. 6 which had also been used for comparison with the analytical results in Ref. 9.) The objective of this phase of the numerical study was to determine whether the present results, which can be obtained with minimal computer usage, compare well with those of the more exact and complex analytical results in Ref. 9. The results of this comparison are highly encouraging, as will be shown below, and support the utilization of the present analysis as an efficient design tool.

In the second phase of the design numerical studies, consideration was given to examining the sensitivity of laminate results to individual properties of the layers. These parametric studies are presented for several classes of typical laminates.

A series of laminates of boron/epoxy composites for which experimental data had been obtained in Ref. 6 were examined analytically in Ref. 9. In Figures 9 to 15, results of the present analytical method are added to the comparison of experimental results of [6] and analytical results of [9]. For example, in Fig. 9, the experimental stress-strain curve for a 0-90 boron/epoxy laminate is compared to the analytical results obtained in Ref. 9 and in the present analysis. Both analytical results coincide; both show slightly less inelastic strain than the experiment. The solid point on the curve indicates the stress level at which fiber fracture is computed to occur in one of the layers of the laminate.

The shear stress-strain curve used in the present analysis was the best fit Ramberg-Osgood curve having an exponent $n=3$.

34.

The values of modulus and yield stress obtained from the least squares fit are shown on the figure. A similar result is shown for the unidirectional tension $\pm 45°$ laminate in Fig. 10. Here it is seen that the two analytical curves are similar, although the agreement is not as close as in Fig. 9. Experimental data reflect a substantially higher degree of inelasticity than either analytical result. The present analysis shows a higher degree of inelastic strain at the higher stress level than that of Ref. 9. However, the reverse is true in the comparison of the two analytical results shown in Fig. 11 for a $\pm 30°$ laminate. The present results were obtained with a linear stress-strain curve in the transverse direction within each of the layers. The computations were made in this fashion because the transverse stress-strain curve of Ref. 9 does not show a significant degree of inelasticity.

Figure 12 presents results for the case of a quasi-isotropic laminate $(0/\pm 45/90)$ of boron/epoxy. Both the present result and that of Ref. 9 show a relatively insignificant amount of inelasticity. Again, the experimental data show a greater inelastic effect. Here the predicted failure strain level is in good agreement with the experimental failure strain level; however, there is a significant difference in the failure stress level. A similar result is presented in Fig. 13 for the quasi-isotropic laminate formed from the $0/\pm 60°$ configuration.

Computations performed for the present study for laminates having fibers in several directions, including the loading direction, for a simple unidirectional load have shown a relatively small amount of inelastic strain. Another example of this is presented in Fig. 14 for a $0/\pm 45°$ laminate. Here, however, the agreement of all the analytical methods and the experimental method is very good.

The final comparison taken from Ref. 9 is presented in Fig. 15 for a laminate having fibers in three different directions and a tensile load applied at some intermediate angle. The present analysis agrees reasonably well with the results

35.

Ref. 9. The discrepancy between the failure load predicted on the basis of fiber failure and the experimentally observed failure stress is quite substantial. It is possible that fialure in laminate of this type caould result from shearing or transverse stresses within the individual layers, and thus, not be a result of tension in the fiber failure. This mode of failure has not been treated in the present computer program. The mode of failure observed experimentally is not known to the authors.

The experimentally measured response of a multidirectional laminate to an applied shear stress has been reported in Ref. 13. Comparison of the experimental result with the theory of Ref. 9 was presented in Ref. 14. Computations for this case, made using the present analysis and the prior analytical result (Ref. 14), are compared to the experimental result in Fig. 16. Again, correlation between the two analytical results is good, agreement between analytical and theoretical results is reasonably good with the experimental observation showing higher inelastic strains and lower tangent shear moduli at the very high stress levels.

The conclusion of these comparisons with analytical and experimental data seem to justify the adoption of the present computer program as a useful engineering tool for the design and analysis of composite laminates. However, it appears that further study of the failure region is required.

Parametric study of the influence of various laminate geometric and mechanical properties has also been explored. Fig. 17 shows the results obtained for a 0/±45° laminate indicating that the inelastic response in the transverse direction can become significant at higher stress levels. Failure due to fiber fracture under a transverse stress applied to the laminate occurs at strain levels larger than those plotted in Fig. 17. In the quasi-isotropic laminate having four fiber directions, (0/±45/90) the degree of inelasticity in the longitudinal and transverse directions is of course the same and is

36.

in both cases very small. It is to be anticipated, on an
intuitive basis, that the maximum degree of inelastic response
would be observed for a stress applied midway between two of
the fiber directions on this quasi-isotropic laminate. The
stress-strain curve for this latter case is also shown in Fig.
18. Although the inelastic strains for this case are not
significant there is a large difference in the predicted
failure stress levels based on stress in the fiber direction
for the two cases. It is worthwhile to emphasize that the
quasi-isotropic laminate need not be isotropic in its strength
characteristics.

Because of the directional strength characteristics inter-
esting effects may be expected for combined stress cases. Some
results of the exploration of this question are presented in
Fig. 19 where the four direction quasi-isotropic laminate is
subjected to combined stress state with respect to a 22-1/2°
axis of symmetry. This laminate shows high strength under both
the unidirectional load and shear load by itself. The combined
stress case for equal values of applied shear stress and axial
stress results in fiber failure, and therefore, laminate failure,
at a substantially lower stress. The stress-strain curve prior
to failure is not affected significantly by the presence of com-
bined stress. The quasi-isotropic laminate having fibers in
three directions (0/±60) is examined in Fig. 20. The sensitivity
of this laminate to the Ramberg-Osgood parameters for the indi-
vidual ply had little effect upon the stress-strain result. In-
deed as an extreme example of this variation all laminates stiff-
nesses except the axial stiffness were equal to zero. Enforcement
of the Kirchhoff-Love plate assumptions for this case results
in the so-called netting analysis. The response for this net-
ting case, which is linear, is shown by the dashed curve in
Fig. 20. Even with this extreme assumption, matrix inelasticity
does not introduce a significant amount of inelastic strain.
Experimental data for comparison with this result are not easily
available, however Ref. 17 does present a stress-strain curve
for this case which shows a transverse failure stress for the

37.

quasi-isotropic 0/±60° laminate which is about 60% of the failure stress in the axial direction. Also, the inelastic strain at failure is approximately 30% larger than the elastic strain associated with the failure stress level. The netting analysis result presented here suggests that in order to obtain such a strain, one might have to consider that the axial stiffness, either in tension, compression or both; or that other effects not considered in the conventional laminate analysis, such as interlaminar or transverse shear deformations, might contribute significantly to the overall laminate deformation.

The influence of the characteristic stress levels for transverse stress and axial shear of the unidirectional layer of a boron/epoxy material is examined in Fig. 21. The measure of this effect is taken to be the influence upon the stress-strain curve for the unidirectional tension of ±30° laminate. The strong sensitivity to the characteristic axial shear stress τ_y and the relative insensitivity to the transverse characteristic stress σ_y for the R-O representations is illustrated in the figure. A similar comparison made for a boron/aluminum laminate of the same geometry subjected to uniaxial applied stress is shown in Fig. 22. Similar sensitivities are observed for this case. Boron/aluminum laminate response under transverse applied stress with the same values of the Ramberg-Osgood parameters is shown in Fig. 23. Here the fiber failure criterion did not come into play and thus the computations were extended to rather large strains in matrix. It is clear, that for this case, the failure criterion based on other stress-strain components is required. The examination of the computer print-out permits one to terminate the stress-strain curves at some stress level prior to fiber fracture depending upon the choice of the failure criterion. This can be done rather readily. The choice of the failure criterion is discussed in Appendix D.

The lamina properties for boron/aluminum are used to

38.

analyze a 0°/±30° laminate under combined loading. These re-
sults are shown in Fig. 24. Axial stress-strain curves are
presented for varying ratios of axial shear stress to axial
tensile stress.

4. CONCLUDING REMARKS

Current approaches to the definition of design allowable
stress for advanced fiber composite laminates are based upon
the utilization of extremely conservative criteria. These limit
the laminate to stress levels below which no significant damage
of any kind occurs. The utilization of overly conservative de-
sign criteria can negate much of the potential for effective de-
sign utilizing advanced composite materials. The heterogeneous
nature of these materials is such that a variety of possible
damage modes exist. Thus, matrix cracking or yielding, fiber
fracture, debonding, and other inelastic effects can all occur
in local regions at relatively low average stress levels. These
nonuniform and nonlinear effects greatly complicate the problem
of establishing reliable design allowables. In the present pro-
gram, the problem of nonlinear laminate behavior resulting from
nonlinearities in the behavior of the matrix material was studied.
The objective of the program was to develop an understanding of
the inelastic behavior of composite laminates and to develop a
computer program which will be used as an engineering tool in
the design of fiber composite laminated structures.

The method of approach utilized herein was to adopt a Ram-
berg-Osgood representation of the nonlinear stress-strain be-
havior and to utilize deformation theory as an adequate repre-
sentation of the material nonlinearities. The problem was viewed
on two levels. First, the relationship between the constituent
properties and the stress-strain response of a unidirectional
fiber composite material was studied. For this problem, the
primary attention herein was directed toward the axial shear be-
havior, in as much as experimental data had indicated that it is
this type of load which results in the most significant nonlineari-
ties in material behavior. For this case, an expression was estab-
lished relating the composite average-stress/average-strain curve
to the fiber moduli and the matrix nonlinear stress-strain curve.
This expression, which was developed as a lower bound, was found
to give good agreement with the more exact results obtained by
40.

applying incremental plasticity theory and using a numerical finite element analysis to the assessment of the material behavior (Ref. 9).

The second level of approach treats the interelationship between the properties of the unidirectional layers and those of the laminate. For this case, one may consider that the starting point is a nonlinear stress-strain curve for transverse stress, and for axial shear stress, alone, and a linear stress-strain relation for stress in the fiber direction. The nonlinear lamina stress-strain curves can be modeled by proper selection of the Ramberg-Osgood parameters.

In the present study, unlike other formulations an interaction expression was formulated to account for simultaneous application of axial shear and transverse stress. A laminate having an arbitrary number of oriented layers, and subjected to a general state of membrane stress, was treated. The results of this analysis were programmed into an efficient computer routine for numerical evaluation of arbitrary laminates. Results obtained show good agreement with those of previous complex numerical methods utilizing incremental plasticity theory.

Certain limitations connected with this program should also be discussed. First, deformation type stress-strain relations have been used; hence, it is implicit in this result that the stress and strain values obtained for any given set of loads are functions only of those loads and not of the loading history. On the other hand, if points are computed for intermediate values of loads, following different load paths, then different intermediate conditions will be obtained. Thus, the question is raised as to what is the accuracy of the results obtained for paths which do not yield proportional loading. It is known that for local proportional loading, the deformation theory result is the same as that for the incremental theory. In the laminate, local proportional loading does not exist, in general, even when the external loading is proportional. However, the assumption is made that the deformation theory will yield an approximation which is satisfactory to generate a

41.

rational engineering tool. This can only be assessed by comparison with an exact analysis, or since this does not exist for the case of arbitrary loading paths, perhaps by comparison with experimental data.

Comparisons of the present results with experimental data tend to show moderately good agreement. There are, however, cases in which experimental results show a higher degree of inelastic strain than predicted by the present analysis. These experimental data are quite limited and may be insufficient for drawing conclusions in this regard.

The question of failure criteria incorporated into the present analysis required further consideration. The present analysis obtains more accurate representations of the stress components in the individual layers than have been obtained from elastic analyses. Hence, the use of these stress components in any failure criteria should represent an improvement in failure prediction

In addition to a description of the methods of analysis, and of the numerical comparisons which have been carried out, the present report also presents a description of the computer program for study of nonlinear behavior of laminates in sufficient detail to permit the utilization of this program by others.

42.

APPENDIX A

SYMMETRY SIMPLIFICATION OF NON-LINEAR STRESS-STRAIN RELATIONS

The most general inelastic stress-strain relations of the deformation type are of the form

$$\epsilon_{ij} = S_{ijkl}\,\sigma_{kl} \tag{1}$$

where S_{ijkl} are functions of the stresses. Let it be assumed that the material is transversely isotropic with x_1 axis of symmetry. Any rotation about x_1 changes ϵ_{ij} and σ_{ij} into ϵ'_{ij} and σ'_{ij}. Then the condition of transverse isotropy demands that

$$\epsilon'_{ij} = S_{ijkl}\,\sigma'_{kl} \tag{2}$$

where S_{ijkl} in (1) and (2) are the same. To fulfill this last requirement it is necessary that S_{ijkl} be functions of stresses only through stress expressions which are invariant for rotations about the x_1 axis. There are five such invariants and they are given by, [15]

$$I_1 = \sigma_{11} \qquad I_2 = \sigma_{22} + \sigma_{33} \qquad I_3 = \sigma_{12}^{\,2} + \sigma_{13}^{\,2} \tag{3}$$

$$I_4 = 1/2\,(\sigma_{22} - \sigma_{33})^2 + 2\sigma_{23}^{\,2} \qquad I_5 = 1/2\,(\sigma_{22} - \sigma_{33})(\sigma_{12}^{\,2} - \sigma_{13}^{\,2}) - 2\sigma_{12}\sigma_{13}\sigma_{23}$$

Thus

$$S_{ijkl} = S_{ijkl} (I_1, I_2, I_3, I_4, I_5) \tag{4}$$

It follows that for rotations around the x_1 axis of symmetry the S_{ijkl} behave as constants. Consequently, the symmetry reduction of (1) to transverse isotropy is just as in elasticity.

The reduction may be performed in following fashion: For rotation of angle θ about the x_1 axis, the stress tensor σ_{ij} transforms into σ'_{ij} in the following fashion

$$\sigma'_{11} = \sigma_{11}$$

$$\sigma'_{22} = 1/2 \, (\sigma_{22} + \sigma_{33}) + 1/2 \, (\sigma_{22} - \sigma_{33}) \cos 2\theta + \sigma_{23} \sin 2\theta$$

$$\sigma'_{33} = 1/2 \, (\sigma_{22} + \sigma_{33}) - 1/2 \, (\sigma_{22} - \sigma_{33}) \cos 2\theta - \sigma_{23} \sin 2\theta$$

$$\sigma'_{23} = 1/2 \, (\sigma_{33} - \sigma_{22}) \sin 2\theta + \sigma_{23} \cos 2\theta$$

$$\sigma'_{12} = \sigma_{12} \cos \theta + \sigma_{13} \sin \theta \tag{5}$$

$$\sigma'_{13} = -\sigma_{12} \sin \theta + \sigma_{13} \cos \theta$$

The same transformation relations obviously also hold for strains. If the transformed stresses and strains are introduced into (2) then coefficients of $\cos 2\theta$, $\sin 2\theta$, $\cos \theta$ and $\sin \theta$ and remaining terms independent of θ must be equal. These equalities result in relations among the various components which reduce the stress-strain law to the form (2.1.4- 5) from Chapter 2 of this report. (Average stresses and strains appear in the latter but this obviously makes no differences in the derivation.)

44.

APPENDIX B

PLANE STRESS-STRAIN RELATIONS OF FIBER REINFORCED MATERIAL IN GENERALIZED RAMBERG-OSGOOD FORM

The purpose of the present appendix is to arrive at equs. (2.2.7). For convenience in writing, overbars on stresses and strains will be omitted.

The present development is guided by isotropic J_2 theory for deformation type plastic stress-strain relations. The basic assumption of this theory in the isotropic case is that the plastic strains have the form

$$\varepsilon_{ij}^{\ast} = f(J_2) s_{ij} \tag{1}$$

where s_{ij} is the stress deviator and

$$J_2 = 1/2 \, s_{ij} \, s_{ij} \tag{2}$$

is its second invariant.

It is instructive to work out the form of (1) for Ramberg-Osgood type stress-strain relations. Suppose that in pure shear the stress-strain relation is

$$\varepsilon_{12}^{\ast} = \frac{\sigma_{12}}{2G} \left[1 + \left(\frac{\sigma_{12}}{\tau_y} \right)^{n-1} \right] \tag{3}$$

Now in pure shear it follows from (2) that

$$J_2 = \sigma_{12}^2$$

Therefore (3) can be written in the form

$$\varepsilon_{12}^{\ast} = \frac{\sigma_{12}}{2G} \left[1 + \left(\frac{\sqrt{J_2}}{\tau_y} \right)^{n-1} \right] \tag{4}$$

which is in the form (1). Consequently, in the general case of three dimensional stress and strain

$$\varepsilon_{ij}^{\ast} = \frac{s_{ij}}{2G} \left[1 + \left(\frac{\sqrt{J_2}}{\tau_y} \right)^{n-1} \right] \tag{5}$$

It should be emphasized that there is nothing fundamental about (1). It is an assumption which states that the plastic strains can be represented by the stress deviator components multiplied by a function of a quadratic expression in the stresses which is J_2. The choice of J_2 for a quadratic expression is not arbitrary but may be arrived at by isotropy arguments.

In an anisotropic material it may be assumed by generalization that plastic strains are given by

$$\epsilon_{ij}^{"} = s_{ij} \, f \, (L) \tag{6}$$

Where L is some general quadratic function of the stresses. This assumption will form the basis of the present development.

Consider the stress-strain relations (2.1.13). It is assumed that $s_{22}^{"}$ and $s_{44}^{"}$ functions of the most general quadratic form in $\bar{\sigma}_{22}$ and $\bar{\sigma}_{12}$. Thus

$$s_{22}^{"} = s_{22}^{"} \quad (A\bar{\sigma}_{22}^{2} + B\bar{\sigma}_{22}\,\bar{\sigma}_{12} + C\bar{\sigma}_{12}^{2})$$

$$s_{44}^{"} = s_{44}^{"} \quad (A\bar{\sigma}_{22}^{2} + B\bar{\sigma}_{22}\,\bar{\sigma}_{12} + C\bar{\sigma}_{12}^{2}) \tag{7}$$

It should be noted that the material reacts in same fashion to positive or negative shear stress, therefore also in same fashion to some $\bar{\sigma}_{22}$ together with positive or negative shear stress. However, the middle term in the quadratic changes sign with shear stress. Therefore, this term should be omitted.

Now rewrite (7) in form

$$s_{22}^{"} = \frac{1}{E_T} \, f_{22} \, (\alpha^2 \bar{\sigma}_{22}^{2} + \beta^2 \bar{\sigma}_{12}^{2})$$

$$s_{44}^{"} = \frac{1}{2G_T} \, f_{44} \, (\alpha^2 \bar{\sigma}_{22}^{2} + \beta^2 \bar{\sigma}_{12}^{2}) \tag{8}$$

46.

where f_{22} and f_{44} are nondimensional functions and α and β have dimensions of reciprocal of stress. If $\bar{\sigma}_{12}=0$ the first of (8) assumes the form

$$s_{22}'' = \frac{1}{E_T} f_{22} (\alpha^2 \bar{\sigma}_{22}{}^2) \tag{9}$$

For one dimensional $\bar{\sigma}_{22}$, from the Ramberg-Osgood stress-strain relation (2.2.6a)

$$s_{22}'' = \frac{1}{E_T} (\frac{\bar{\sigma}_{22}}{\sigma_y})^{M-1}$$

which can be written as

$$s_{22}'' = \frac{1}{E_T} [(\frac{\bar{\sigma}_{22}}{\sigma_y})^2]^{\frac{M-1}{2}} \tag{10}$$

It follows from (8) and (10) that

$$\alpha^2 = \frac{1}{\sigma_y{}^2} \tag{11}$$

and the function of f_{22} is determined as $(M-1)/2$ power. In similar fashion, when $\bar{\sigma}_{22}=0$, the second of (8) assumes the form

$$s_{44}'' = \frac{1}{2G_T} f_{44} (\beta^2 \bar{\sigma}_{12}{}^2) \tag{12}$$

From the Ramberg-Osgood relation (2.2.6b) for one dimensional $\bar{\sigma}_{12}$

$$s_{44}'' = \frac{1}{2G_T} (\frac{\bar{\sigma}_{12}}{\tau_y})^{N-1}$$

which can be written as

$$s_{44}'' = \frac{1}{2G_T} [(\frac{\bar{\sigma}_{12}}{\tau_y})^2]^{\frac{N-1}{2}} \tag{13}$$

It follows from (12) and (13) that

$$\beta^2 = \frac{1}{\tau_y{}^2} \tag{14}$$

and the function f_{44} is determined as $(N-1)/2$ power.

Consequently (8) now assumes the form

$$s_{22}'' = \frac{1}{E_\tau} \left[\left(\frac{\bar{\sigma}_{22}}{\sigma_y}\right)^2 + \left(\frac{\bar{\sigma}_{12}}{\tau_y}\right)^2 \right] \frac{M-1}{2}$$

$$s_{44}'' = \frac{1}{2G_\tau} \left[\left(\frac{\bar{\sigma}_{22}}{\sigma_y}\right)^2 + \left(\frac{\bar{\sigma}_{12}}{\tau_y}\right)^2 \right] \frac{N-1}{2}$$

(15)

Then (2.2.7) follows from (15) and (2.1.13).

48.

APPENDIX C

1. ## EXTREMUM PRINCIPLES OF DEFORMATION THEORY OF PLASTICITY

   i. ## Principle of Minimum Potential Energy

   Let

   $$c_{ij} = C_{ijkl}\, \epsilon_{kl} \qquad (1.1)$$

   where C_{ijkl} are functions of the strains. The strain energy density is defined by the path dependent integral

   $$w^\epsilon = \int_{\epsilon=0}^{\epsilon} c_{ij}(\underline{\epsilon})\, d\epsilon_{ij} \qquad (1.2)$$

   where $\underline{\epsilon}$ is a concise notation for ϵ_{ij}. The strain energy U^ϵ of a body of volume V is defined by

   $$U^\epsilon = \int_V w^\epsilon\, dV \qquad (1.3)$$

   Let the surface of the body be subjected to the boundary conditions

   $$u_i(S) = u^\circ_i \quad \text{on } S_u \qquad (1.4)$$

   $$T_i(S) = T^\circ_i \quad \text{on } S_T$$

   and let the body forces vanish. The potential energy U_p is defined by

   $$U_p = \int_V w^\epsilon\, dV - \int_{S_T} T^\circ_i u_i \qquad (1.5)$$

   Define an admissible displacement field $\tilde{u}_i(\underline{x})$ by

   $$\tilde{u}_i = u^\circ_i \quad \text{on } S_u$$

   $$\tilde{u}_i(\underline{x}) \text{ continuous everywhere} \qquad (1.6)$$

   Associated with $\tilde{u}_i$ are the strains $\tilde{\epsilon}_{ij}$ derived from it by the usual relations.

Define $\tilde{W}^s$ by

$$\tilde{W}^s = \int_{\tilde{\varepsilon}=0}^{\tilde{\varepsilon}} \tilde{\sigma}_{ij} \, d\tilde{\varepsilon}_{ij} \qquad (1.7)$$

where

$$\tilde{\sigma}_{ij} = C_{ijkl}(\tilde{\varepsilon}) \, \tilde{\varepsilon}_{kl} \qquad (1.8)$$

Define

$$\tilde{U}_p = \int_V \tilde{W}^s \, dV - \int_{S_T} T^0_i \, \tilde{u}_i \, dS \qquad (1.9)$$

The principle of minimum potential energy for the present case then states that

$$\tilde{U}_p \geq U_p \qquad (1.10)$$

equality taking place if and only if

$$\tilde{u}_i = u_i$$

In the event that displacements are prescribed over the entire surface, the surface integral in (1.9) vanishes. Then the principle reduces to that of minimum strain energy

$$\tilde{U}^s \geq U^s \qquad (1.11)$$

ii. **Principle of Minimum Complementary Energy**

Let

$$\varepsilon_{ij} = S_{ijkl}(\underline{\sigma}) \, \sigma_{kl} \qquad (1.12)$$

where S_{ijkl} are stress dependent compliances

50.

Define the complementary energy density w^c by the path dependent integral

$$w^c = \int_{\underline{\sigma}=0}^{\underline{\sigma}} \varepsilon_{ij} d\sigma_{ij} \qquad (1.13)$$

Let the surface of the body be subjected to the boundary conditions (14) and let the body forces vanish. The complementary energy U_C is defined by

$$U_C = \int_V w^c dV - \int_{S_u} T_i u^\circ_i \, dS \qquad (1.14)$$

Define an admissible stress field $\tilde{\sigma}_{ij}$ by the following requirements

$$\tilde{\sigma}_{ij'j} = 0$$

$$\tilde{T}_i = \tilde{\sigma}_{ij} n_j \qquad \text{continuous everywhere} \qquad (1.15)$$

$$T_i(S) = T^\circ_i \text{ on } S_T$$

Define the complementary energy functional $\tilde{U}_C$ by

$$\tilde{U}_C = \int_V \tilde{w}^c dV - \int_{S_u} \tilde{T}_i u^\circ_i dS \qquad (1.16)$$

where
$$\tilde{w}^c = \int_{\tilde{\sigma}=0}^{\tilde{\sigma}} \tilde{\varepsilon}_{ij} d\tilde{\sigma}_{ij} \qquad (1.17)$$

$$\tilde{\varepsilon}_{ij} = S_{ijkl}(\tilde{\sigma}) \tilde{\sigma}_{kl}$$

Then the principle of minimum complementary energy states that

$$\tilde{U}_C \geq U_C \qquad (1.18)$$

equality occurring if and only if

$$\tilde{\sigma}_{ij} = \sigma_{ij}$$

If tractions are prescribed over the entire surface, $S_u = 0$, the principle reduces to

$$\tilde{U}^\sigma \geq U^c \qquad (1.19)$$

51.

For proof of these principles see e.g. [16]. An interesting application to obtain approximate solutions has been given in [17].

iii. Specialization of the Principles to Axial Shear with Ramberg-Osgood Stress-Strain Relations

In the case of axial shear of a uniaxially fiber reinforced material the only surviving stresses are

$$c_{12} = \tau_2 \qquad c_{13} = \tau_3 \qquad\qquad (1.20)$$

where 1 indicates fiber direction. Denote the associated shear strains by

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad (1.21)$$
$$\varepsilon_{12} = \varepsilon_2 \qquad \varepsilon_{13} = \varepsilon_3$$

Then the generalized Ramberg-Osgood stress-strain relations, Appendix B, (5) assume in the present case the form

$$\varepsilon_2 = \frac{\tau_2}{2G} \left[1 + \left(\frac{\tau}{\tau_y}\right)^{n-1} \right]$$

$$\varepsilon_3 = \frac{\tau_3}{2G} \left[1 + \left(\frac{\tau}{\tau_y}\right)^{n-1} \right]$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad (1.22)$$

$$\tau = \sqrt{\tau_2^2 + \tau_3^2} \qquad \sqrt{\tau_2}$$

In the present case

$$c_{ij} d\varepsilon_{ij} = 2(\tau_2 d\varepsilon_2 + \tau_3 d\varepsilon_3) \qquad\qquad (1.23)$$

52.

Inserting (1.22) into (1.23) and using the relation

$$\tau d\tau = \tau_2 d\tau_2 + \tau_3 d\tau_3$$

it is easily shown that

$$\sigma_{ij} d\varepsilon_{ij} = \frac{\tau}{G} \left[1 + n \left(\frac{\tau}{\tau_y}\right)^{n-1} \right] d\tau \qquad (1.24)$$

To compute W^ε as defined by (1.2) it is necessary to integrate (1.24) from zero to some state of strain ε_2, ε_3. But it should be noted that (1.24) is expressed in terms of the variable τ only. Now τ can be expressed in terms of strains in following fashion. Define

$$\hat{\varepsilon} = \sqrt{\varepsilon_2^2 + \varepsilon_3^2} \qquad (1.25)$$

It follows at once from (1.22) that

$$\hat{\varepsilon} = \frac{\tau}{2G} \left[1 + \left(\frac{\tau}{\tau_y}\right)^{n-1} \right] \qquad (1.26)$$

This relation defines τ as a function of $\hat{\varepsilon}$. Consequently, W^ε assumes the form

$$W^\varepsilon = \frac{1}{G} \int_o^{\tau(\hat{\varepsilon})} \tau \left[1 + n \left(\frac{\tau}{\tau_y}\right)^{n-1} \right] d\tau$$

which is easily integrated to yield

$$W^\varepsilon = \frac{\tau^2}{2G} \left[1 + \frac{2n}{n+1} \left(\frac{\tau}{\tau_y}\right)^{n-1} \right] \qquad (1.27)$$

$$\tau = \tau(\hat{\varepsilon})$$

According to (1.3) the strain energy U^ε is then given by the volume integral of (1.27). Note however that it is very difficult to express U^ε in terms of strains since this requires the solution of (1.26) for τ in terms of $\hat{\varepsilon}$. In general it is not possible to do this analytically. This places a severe

limitation on the use of the principle of minimum potential
energy or of minimum strain-energy with Ramberg-Osgood stress-
strain relations.

Next we consider the principle of minimum complementary
energy for axial shear. Since there are only shear stresses
τ_2, τ_3 and shear strains ϵ_2, ϵ_3 the integrand in W^σ,
(1.13), is given by

$$\epsilon_{ij}d\sigma_{ij} = 2(\epsilon_2 d\tau_2 + \epsilon_3 d\tau_3) \tag{1.28}$$

It follows from (1.22-.23) that (1.28) is given by

$$\epsilon_{ij}d\sigma_{ij} = \frac{\tau}{G}\{1 + (\frac{\tau}{\tau_y})^{n-1}\}\,d\tau$$

Integration of this expression from 0 to τ yields

$$W^\sigma = \frac{\tau^2}{2G}\{1 + \frac{2}{n+1}(\frac{\tau}{\tau_y})^{n-1}\} \tag{1.29}$$

Expression (1.29) now enters as the integral into the volume
integral of U_c, (1.14).

We now examine the meaning of an admissible stress field
$\tilde{\tau}_2$, $\tilde{\tau}_3$ in the present case. The only surviving equilibrium
equation is

$$\frac{\partial\tilde{\tau}_2}{\partial x_2} + \frac{\partial\tilde{\tau}_3}{\partial x_3} = 0 \tag{1.30}$$

The traction components are

$$\tilde{T}_1 = \tilde{\tau}_2 n_2 + \tilde{\tau}_3 n_3$$

$$\tilde{T}_2 = \tilde{\tau}_2 n_1 \tag{1.31}$$

$$\tilde{T}_3 = \tilde{\tau}_3 n_1$$

We shall be concerned with cylindrical boundaries in fiber
reinforced materials whose generator is in x_1, direction. On such
a surface $n_1 = 0$. Therefore the only surviving traction compo-
nent on such a surface is

$$\tilde{T}_1 = \tilde{\tau}_n = \tilde{\tau}_2 n_2 + \tilde{\tau}_3 n_3 \tag{1.32}$$

54.

Consequently an admissible stress system $\tilde{\tau}_3$, $\tilde{\tau}_3$ must satisfy (1.30) and the value τ°_n of $\tilde{\tau}_n$ wherever prescribed on the boundary.

The complementary energy functional (1.16) assumes the form

$$\tilde{U}_C = \int_V \tilde{W}^\sigma dV - \int_{S_u} \tilde{\tau}_1 u^\circ_1 dS \qquad \text{(a)}$$

$$\tilde{W}^\sigma = \frac{\tilde{\tau}^2}{2G} \left[1 + \frac{2}{n+1} \left(\frac{\tilde{\tau}}{\tilde{\tau}_y} \right)^{n-1} \right] \qquad \text{(b)}$$

$$\hfill (1.33)$$

$$\tilde{\tau} = \sqrt{\tilde{\tau}_2^2 + \tilde{\tau}_3^2} \qquad \text{(c)}$$

55.

2. LOWER BOUND FOR AXIAL SHEAR MODULUS

Consider a uniaxially reinforced lamina which is subjected to axial shear τ_0 in the 1-2 plane on its boundary, fig. 5.

By the average stress theorem, of Ref. 5.

$$\bar{\sigma}_{12} = \tau_0 \tag{2.1}$$

and all other average stresses vanish.

By the average theorem of virtual work, of Ref. 5,

$$\int_V \epsilon_{ij} d\sigma_{ij} = \bar{\epsilon}_{ij} d\bar{\sigma}_{ij} \tag{2.2}$$

Since the only nonvanishing average stress in the present case is (2.1) we have

$$\bar{\epsilon}_{ij} d\bar{\sigma}_{ij} = 2\bar{\epsilon}_{12} d\tau_0 \tag{2.3}$$

The complementary energy of the body is given by (14) of Appendix A. The surface integral vanishes however in the present case since no displacements are prescribed on the boundary. Now

$$U_C = \int_V w^c dV = \int_V \int_{c=0}^{\sigma} \bar{\epsilon}_{ij} d\bar{\sigma}_{ij} dV \tag{2.4}$$

$$= \int_{c=0}^{\sigma} \int \bar{\epsilon}_{ij} d\bar{\sigma}_{ij} dV \int_0^{\tau_0} \bar{\epsilon}_{12} d\tau_0$$

The last equality following from (2.2, 3).

By definition the effective secant modulus G_A^s is given by

$$\bar{\epsilon}_{12} = \frac{\bar{\sigma}_{12}}{G_A^s(\bar{\sigma}_{12})} = \frac{\tau_0}{2G_A^s(\tau_0)} \tag{2.5}$$

Hence (2.4) assumes the form

$$U_C = V \int_0^{\tau_0} \frac{\tau_0 d\tau_0}{G_A^s(\tau_0)} \tag{2.6}$$

In order to find a bound on G_A^s it will be necessary to find a bound on (2.6) by use of the principle of minimum complementary energy.

56.

It is assumed that the fibers are infinitely rigid in comparison to the matrix. Therefore at fiber/matrix interface

$$u_1 = 0 \tag{2.7}$$

and the only contribution to the complementary energy is from the matrix. Thus, the surface integral in (1.33a) vanishes and it can be written as

$$\tilde{U}_c = \int_{V_m} \tilde{W}^\sigma \, dV \tag{2.8}$$

where V_m is the matrix volume.

Furthermore, by (2.3.3) the actual stresses are functions of x_2, x_3 only. It is therefore natural to also choose admissible stresses as functions of x_2, x_3. Thus $\tilde{W}^\sigma$ in (1.33) becomes a function of x_2, x_3 only and therefore without loss of generality (1.33a) can be taken over unit length in fiber direction. Thus it can be written

$$\tilde{U}_c = \int_{A_m} \tilde{W}^c \, (x_2, \, x_3) \, dx_2 \, dx_3 \tag{2.9}$$

In order to construct an admissible stress system it is necessary to devise a geometrical model for a uniaxially reinforced material . In past analyses of FRM two kinds of models have been successfully treated: Periodic arrays of identical circular fibers have been analyzed numerically with the aid of computers and the composite cylinder assemblage model has been treated analytically [1,5] yielding simple closed results. Since the present treatment is to be analytical the composite cylinder assemblage model will be used. A detailed description of the model has been given in[5]. Suffice it to say here that the model represents a cylindrical specimen of a fiber reinforced material as an assemblage of composite cylinders of different sizes which fill the space in the limit. In each composite cylinder the inner cylinder is a fiber and the outer shell is matrix material.

In all cylinders the ratios of fiber to matrix shell radius are the same, (figure 26).

It is recalled that an admissible stress system must satisfy equilibrium and boundary conditions. An obvious possibility for such an admissible field are the stresses of the elastic solution since they certainly satisfy the required conditions. These stresses are the same in any composite cylinder of the assemblage and are given in cylindrical coordinates by (see [5])

$$\bar{\sigma}_{rz} = \bar{\tau}_{r} = \frac{\tau_0}{1+c} (1 + \frac{a^2}{r^2}) \cos \theta \tag{2.10}$$

$$\bar{\sigma}_{\theta z} = \bar{\tau}_{\theta} = - \frac{\tau_0}{1+c} (1 - \frac{a^2}{r^2}) \sin \theta$$

where c is the volume fraction of fibers, a is the radius of any fiber and r, θ are polar coordinates, fig. 26.

Since $\bar{\tau}$ as expressed by (1.33c) is an invariant with respect to rotations about $x, = z$ we have also

$$\bar{\tau}^2 = \bar{\tau}_r^2 + \bar{\tau}_\theta^2 \tag{2.11}$$

Substituting (2.10) into (2.11) yields

$$\bar{\tau}^2 = p^2 \ (1 + \frac{1}{\zeta^4} + \frac{2}{\zeta^2} \cos \theta) \tag{2.12}$$

where

$$p = \frac{\tau_0}{1+c} \qquad \qquad \zeta = \frac{r}{a} \tag{2.13}$$

To simplify the analysis the exponent n in (1.22) will be assigned the value

$$n = 3 \tag{2.14}$$

It has been found that with this value of n, experimentally obtained shear stress-strain relations of epoxy can be quite accurately represented with proper choice of τ_y. Recalling (1.33), (2.9) then assumes the form

$$\dot{U}_C = \frac{1}{2G} \int_{A_m} \bar{\tau}^2 \ [1 + \frac{1}{2} \ (\frac{\bar{\tau}^2}{\tau_y})^2] \ dA \tag{2.15}$$

58.

where G is the matrix elastic shear modulus. Let the assemblage consist of K composite cylinder. Define $\tilde{U}_C^k$ for the kth composite cylinder by

$$\tilde{U}_C^k = \frac{1}{2G} \int_{A_{mk}} \bar{\tau}^2 \left[1 + \frac{1}{2} \left(\frac{\bar{\tau}}{\tau_y} \right)^2 \right] \, dA \qquad (2.16)$$

where A_{mk} is the matrix area $a_k \le r \le b_k$ in the kth composite cylinder. Then

$$\tilde{U}_C = \sum_{k=1}^{K} \tilde{U}_C^k \qquad (2.17)$$

Since $\bar{\tau}^2$ has been expressed in polar coordinates, (2.12), it is convenient to also evaluate (2.16) in the same coordinates. Using the variable ρ we have

$$\tilde{U}_C^k = \frac{1}{2G} \int_1^\beta \int_0^{2\pi} \bar{\tau}^2 \left[1 + \left(\frac{\bar{\tau}}{\tau_y} \right)^2 \right] \rho \, d\rho \, d\theta \qquad (2.18)$$

where

$$\beta = b_k / a_k \qquad (2.19)$$

which by construction has the same value in all composite cylinders. Note also that the volume fraction of fibers c is given by

$$c = \left(\frac{a_k}{b_k} \right)^2 = \frac{1}{\beta^2} \qquad (2.20)$$

Substituting (2.12) into (2.18) and carrying out the integration we have

$$\tilde{U}_C^k = \frac{\pi b_k^2}{2G} \tau_0^2 \left[\frac{1-c}{1+c} + \left(\frac{\tau_0}{\tau_y} \right)^2 \frac{3+10c-12c^2-c^4}{6(1+c)^4} \right] \qquad (2.21)$$

59.

where (2.20) has been used. It is seen that τb^2_k is the area of the cross section of the kth composite cylinder and the parenthesis has the same value for all composite cylinders. Therefore, if (2.21) is inserted into (2.17) we find

$$\dot{U}_C = \frac{A}{2\bar{G}} \, \tau_o^2 \, [\frac{1-c}{1-c} + (\frac{\tau_o}{\tau_y})^\prime \, \frac{3-10c-12c^2-c^4}{6(1+c)^\prime}]$$ (2.22)

Let (2.22) be written

$$\dot{U}_C = A \int_o^{\tau_o} \frac{1}{A} \, \frac{d\tilde{U}_C}{d\tau_o} \, d\tau_o$$ (2.23)

Without loss of generality (2.6) can be evaluated for unit height of cylindrical specimen. Thus

$$U_C = A \int_o^{\bar{\tau}_o} \frac{\tau_o d\tau_o}{G_A^S(\tau_o)}$$ (2.24)

Now introduce (2.23) and (2.24) into the minimum complementary inequality (1.18). Thus

$$\int_o^{\tau_o} [\frac{1}{A} \, \frac{d\tilde{U}_C}{d\tau_o} - \frac{\tau_o}{G_A^S(\tau_o)}] \, d\tau_o \geq 0$$ (2.25)

Since the integral is positive for all values of τ_o the integrand must also be positive for all values of τ_o. It follows that

$$G_A^S(\tau_o) \geq \frac{A\tau_o}{dU_C/d\tau_o} = G_A^S(-)$$ (2.26)

where the extreme right denotes lower bound on the secant modulus G_A^S. Substituting (2.22) into (2.26) and rearranging we find the lower bound (2.3.9) of Chapter 2.

There naturally arises the question of the establishment of an upper bound. The difficulties involved have been discussed above: It is not in general possible to solve Ramberg-Osgood relations for stresses in terms of strains. It is

60.

therefore not possible to analytically express the potential energy functional in terms of admissible strains.

A possibility to resolve the difficulty is to write inelastic stress-strain relations of type (1.22) in the form

$$\tau_2 = 2G\varepsilon_2 \ [1 - (\frac{\varepsilon}{\varepsilon_y})^{\alpha-1}]$$

$$\tau_3 = 3G\varepsilon_3 \ [1 - (\frac{\varepsilon}{\varepsilon_y})^{\alpha-1}] \qquad\qquad (2.27)$$

$$\varepsilon = \sqrt{\varepsilon_2{}^2 + \varepsilon_3{}^2}$$

where α and ε_y are to be determined by curve fitting. The minus sign in the parenthesis is due to the fact that the stress-strain curve is below a straight line with the initial slope.

It should be noted that (2.27) are not an inversion of (1.22). They are merely another form of approximation of actual stress-strain curves.

In principle the representation (2.27) can now be used in conjunction with the principle of minimum potential energy to establish an upper bound on G_A^s in same fashion as a lower bound has been established. It has however been found that in attempting to fit (2.27) to actual epoxy stress-strain curves a fractional exponent α was needed. This led to integrals of formidable difficulty in the evaluation of potential energy functionals. Therefore this approach has not been continued here.

61.

APPENDIX D

FAILURE OF NON-LINEAR LAMINATES

It is expedient to separate the problem of the establishment of failure criteria of laminates into two separate problems:

(a) Establishment of failure criteria for uniaxially fiber reinforced material, i.e., laminae.

(b) Establishment of failure criteria of the laminate on the basis of laminae failure criteria.

A great deal of wrok has been done on problem (a). The problem has been approached in micro as well as macro-fashion. In micro-approach, it is attempted to predict failure on the basis of local analysis of the interior of the composite. Such an approach evidently encounters extreme difficulties. Although important work of fundamental nature has been done in this area, we shall not be concerned with it here since the work has not advanced to the stage of prediction of failure criteria under states of combined stress.

In the macro-approach, a failure criterion is heuristically postulated as some function of pertinent state variables (generally average stresses) which also contains undetermined parameters. These parameters are then to be determined in terms of experimentally accessible information.

We shall in the present discussion limit ourselves to states of plane stress. The simplest failure criterion is the so-called maximum stress criterion which states that failure occurs when either one of: stress in fiber direction, stress transverse to fibers, shear stress, reaches its critical value, these critical values being the same whether or not the stresses act simultaneously. In symbols the criterion is:

$$\sigma_{11} = \sigma_A$$

or

$$\sigma_{22} = \sigma_T \tag{1}$$

62.

or

$$\sigma_{12} = \tau_{AT}$$

where 1 is fiber direction and 2 is the transverse direction.

Generally, failure stresses σ_A and σ_T are different in tension and compression. This is known as Bauschinger effect. There is evidently no Bauschinger effect for the shear stress. The simplest generalization of (1) to account for Bauschinger effect would be to assume as failure criterion:

$$\sigma_{11} = \sigma_A^+ \quad \text{if} \quad \sigma_{11} > 0$$

$$\sigma_{11} = \sigma_A^- \quad \text{if} \quad \sigma_{11} < 0$$

$$\sigma_{22} = \sigma_T^+ \quad \text{if} \quad \sigma_{22} > 0 \tag{2}$$

$$\sigma_{22} = \sigma_{22}^- \quad \text{if} \quad \sigma_{22} < 0$$

$$\sigma_{12} = \tau_{AT} \quad \text{all} \quad \sigma_{12}$$

whichever occurs first, where (+) and (-) superscripts denote failure stresses in tension and compression respectively. The main drawback of these simple criteria is in that they take no account of interaction effects.

The most commonly used criterion which takes into account interaction is of quadratic form. For plane stress it has the form

$$A_{11}\sigma_{11}^2 + A_{22}\sigma_{22}^2 + A_{12}\sigma_{11}\sigma_{22} + A_{44}\sigma_{12}^2 = 1 \tag{3}$$

Here, products of shear stress with normal stress have been omitted since the material cannot distinguish between positive and negative shear stress. Therefore, odd powers (one, in this case) of shear stress cannot appear.

Applying (3) to failure for stress in fiber direction alone, stress transverse to fiber direction alone, shear stress alone, in turn, it is seen at once that

63.

$$A_{11} = \frac{1}{\sigma_A^2}$$

$$A_{22} = \frac{1}{\sigma_T^2}$$

$$A_{44} = \frac{1}{\tau_{12}^2}$$
(4)

The coefficient A_{12} is troublesome since its determination requires a failure experiment under combined stress. Several authors have proposed to use failure experiments on off-axis specimens under uniaxial stress for the determination of A_{12}. See e.g. [18] for discussion.

The situation becomes more complicated if it is required to take into account Bauschinger effect, that is difference of failure stresses in tension and compression. One possibility to account for this effect is to assume that A_{11}, A_{22} assume different values for tension and compression. The situation regarding A_{12}, however, becomes very awkward as it would have to assume four different values to account for four different possibilities of sign combination in biaxial stressing
and

It is also possible to add linear terms to (3) in which case it would assume the form:

$$A_{11}\sigma_{11}^2 + A_{22}\sigma_{22}^2 + A_{12}\sigma_{11}\sigma_{22} + A_{44}\sigma_{12}^2 +$$
(5)

$$B_1\sigma_{11} + B_2\sigma_{22} = 1$$

Such a device was suggested by Hoffman [19]. In this case it is possible to determine values of A_{11}, B_1, A_{22}, B_2 to account for different tensile and compressive uniaxial failure stresses in fiber direction and transverse to it. But the difficulty of assigning four different values to A_{12} remains, unfortunately.

In summary, the status of quadratic failure criteria has to date not been finalized. However, special versions of such criteria have been successfully fitted to experimental data.

It is of importance to realize that in the fiber reinforced materials used in practice failure predictions on the

64.

basis of maximum stress criterion or quadratic failure criterion are not very different. This is due to the large ratios between strength in fiber direction and transverse and shear strengths and is easiest realized by considering the failure criteria as surfaces in σ_{11}, σ_{22}, σ_{12} stress space. The maximum stress criterion is a very elongated rectangular parallelepiped while the quadratic failure criterion is an ellipsoid. For $A_{12}=0$, Fig. 25 shows this schematically on a cut in the σ_{11}, σ_{22} plane. Thus it is seen that stress points on the two failure surfaces are close together for most parts of the surfaces.

The situation would be entirely different for a material in which σ_A, and σ_T were of comparable magnitudes.

We shall now consider problem (b) i.e., the establishment of laminate failure criteria in terms of laminae failure criteria. The most conservative laminate failure criterion is to assume that once any lamina has failed the laminate has reached its ultimate load. There are cases of laminates in which all laminae would fail simultaneously and then this criterion would be justified. For example: a $\pm\theta$ laminate in which the external load direction bisects the angle between the fibers.

In most cases, however, a certain group of laminae will fail first and failure of remaining groups would require further increase of load. Therefore a more realistic alternative is to determine the load at which the first laminae group fails. At this state, the further carrying capacity of the laminate may be assumed to be given by the remaining undamaged laminae. The increase in load which fails another group of laminae is then determined. This process is continued until failure of all laminae has taken place.

Still another possibility is to assume that when a lamina has failed, certain of its stiffnesses reduce to zero. For example: suppose that a lamina or group of laminae has failed in shear. Such a failure implies a crack through the lamina in fiber direction. In that event, it is reasonable to assume

65.

that the shear and transverse stiffnesses of the lamina are
zero, but it still retains its stiffness in fiber direction.
If, however, a lamina fails because of the stress in fiber
direction the damage is so widespread that all of its stiff-
nesses will be negligible. According to the type of failures
encountered analysis is continued for the damaged laminate
with the new stiffness rearrangement. This process is con-
tinued until failure of all laminae has taken place. This
method of analysis seems to be the most realistic but is
also the most complicated.

In almost all of the practical strength analyses of
laminates in the literature, according to any of the methods
outlined above, the stresses used for failure criteria have
been determined on the basis of elastic laminate analysis.
With the present inelastic laminate analysis, more realistic
stresses are available in a better assessment of laminate
failure loads.

APPENDIX E

MSC-NOLIN COMPUTER PROGRAM

1.   General Description of the Program

This is a computer program developed for the inelastic
analysis of a laminate subject to any constant, arbitrary
combination of in-plane loading. Details of the method of
analysis and of the numerical solution, using the Newton-
Raphson method, have been described in the body of this
report. The essential features of the program are summari-
zed below.

The primary capability of MSC-NOLIN is to compute lam-
inae properties when the laminate loads are defined. There
is also a limited capability to work with constituent proper-
ties, rather than laminae properties, as the input. Details of
the input options are discussed subsequently. Basically, the
inputs required are the stress-strain characteristics of the
individual laminae for each of the three in-plane stress
components applied separately. The stress-strain curves for
transverse stress and for stress and for axial shear stress
are defined by Ramberg-Osgood stress-strain curves. The
parameters for these curves along with the laminae elastic
constants are the required material property inputs.

It has been observed that axial shear stresses in indivi-
dual laminae are a major, perhaps the major, source of non-
linearities in laminate response. Therefore, several additional
options have been included in the MSC-NOLIN to accomodate more
detailed characterization of shear response. First, the lam-
inae shear stress-strain response may be input in tabular
form and a least squares fit to the data is automatically ob-
tained for the R-O yield stress (limited to the use of an
exponent, n=3). Secondly, the matrix shear stress-strain curve
can be input along with fiber elastic properties and the laminae
shear stress-strain curve will be computed. In this latter case,
the laminae elastic constants are also computed.

67.

The input specifies one of two options for the determination of the initial set of stresses to be used in the iteration at each value of applied load on the laminate. In one case the stresses found at one load are increased to the load for which the stresses were evaluated. In the other, and generally used option, the increment between the initial stresses used at the nth laminate load value and the actual stresses found for the (n-1)st load value bears the same relation to the ratio of those two load values as the similar relation computed at the previous load cycle, that is,

$$\frac{\sigma_{ij}^{(n)} - \sigma_{ij}^{(n-1)}}{F_n/F_{n-1}} = \frac{\sigma_{ij}^{(n-1)} - \sigma_{ij}^{(n-2)}}{F_{n-1}/F_{n-2}}$$

The program contains a number of controls to define: the size and number of steps of loading at which computations are made; the maximum number of iterations to be permitted in the numerical solution; the desired accuracy to be obtained in convergence; the criteria for divergence of the solution in the iterative process to avoid the use of unnecessary execution time in the case of breakdown of the solution procedure. The program defines the failure of the laminate in a limited fashion, either on the basis of the maximum allowable stress in the fiber in tension or compression, or on the basis that the tangent modulus of the stress-strain curve of the laminate becomes less than a specified value. Failure due to shear or transverse stress are not included at this stage in the development of the program.

2. **Input**

The main features of input in this program are the following:

    (a)   Specify the number of laminates or problems to be solved;

68.

(b) Define the geometrical properties of each layer;

(c) Define either the material properties of each layer or the properties of its constituents;

(d) Define either of the following for each layer:

    (i) yield stress in transverse direction and yield stress in shear;

    (ii) yield stress in transverse direction and a table of values defining shear stress-strain curve for the matrix plus a set of values of stresses to be used for the computation of yield stress in shear;

(e) Specify the type of Ramberg-Osgood relation to be used;

(f) Define the loadings; and

(g) Define the control parameters.

A guide to the preparation of input data for this program is given in section 4 below.

3. <u>Details of Output</u>

The output can be divided basically into two steps:

(a) Output of Input Data:

The first section of the output deals with the output of the input data. If the input is in the form of properties of constituents of the layer, it gives an output of the properties of the constituents first and then the computed value of the properties of the layer; otherwise, it gives output directly the properties of the layer.

(b) Output of Stresses and Strains:

For each set of loading, the computer prints the following:

    (1) value of the load applied;

    (2) number of iterations for convergence;

    (3) stresses for individual laminae with respect to principal elastic axes of the laminae; and

    (4) strains for individual laminae in terms of both laminae and laminate axes.

4. Input Details for MSC-NOLIN

(1) Read (I5) NSETS

    NSETS: number of problems

(2) Read (I5) LAY

    LAY: number of layers in this laminate analysis

(3) Read (I5) INP

    INP: Option for reading in material properties

    INP = 1; read in material properties of individual
laminae;

    INP = 2; compute properties of laminae from the
properties of constituents.

(4) (a) If INP = 1

    (i) Read (5D15.5) $E_{11}, E_{22}, \nu_{12}, \nu_{21}$

    (ii) Read (5D15.5) G_{12}, SY, TY

    (iii) Read (D15.5,I5) T, IANG

  (b) If INP = 2

    (i) Read (4D15.5) EF, MUF, GF, VF

    (ii) Read (3D15.5) EM, MUM, G..

    (iii) Read (I5) I2

        If I2 = 0; read in SY and TY
        (i) Read (2D15.5) SY, TY

        If I2 = 1; TY is to be computed
        (i) Read (5,1002) SYCE
        (ii) Read (2 I 5) NUMT
            NUMT = number of values in the table
        (iii) Read (5D15.5) TAU (J), J=1, NUMT
            (Table of shear stress values of
            matrix read in)
        (iv) Read (5D15.5) GAM (J),J=1, NUMT
            (Table of shear strain values of
            matrix read in)
        (v) Read (5D15.5) SG12 (J), J=2,11
            (Table of shear stress values of
            laminae read in)

(5) Read (5D15.5) XN, XM

XN: exponent in nonlinear transverse stress-
strain law;

XM: exponent in nonlinear shear stress-strain
law.

(6) Read (5D15.5) SO11, SO22, SO12

SO11: applied stress in X-direction
SO22: applied stress in Y-direction
SO12: shear stress in XY

(7) Read (I5, D15.5) KSGM, SMLT

KSGM: total number of loading increments

SMLT: ratio of load increment to the initial
load.

(8) Read (D15.5) STIFF

STIFF: tangent modulus of stress-strain curve
in terms of the laminate axes; specify
a value of STIFF below which the program
will not run.

Read (D15.5) SGR

SGR: maximum allowable stress in the fiber in
tension or compression

(9) Read (I 5, 2D15.5) IT, EPS, UPBD

IT: maximum number of iteration permitted in
Newton-Raphson analysis

EPS: convergence criteria; (ratio of values of
two successive iterations should be less
than EPS)

UPBD: divergence criteria (solution will stop
if ratio of two successive iterations is
greater than $10^{\pm 12}$)

(10) Read (I5) INMT

If INMT = 1, the program uses ratio of previous
two solutions as the initial guess
value iteration process;

If INMT = 2, the program uses extrapolated value
of previous two solutions proportioned
on the basis of stress ratio as the
initial guess.

71.

REFERENCES

1. Z. Hashin and B.W. Rosen - The Elastic Moduli of Fiber Reinforced Materials - J. Appl. Mech. 31, 223, (1964).

2. G. Pickett - Elastic Moduli of Fiber Reinforced Plastic Composites - in Fundamental Aspects of Fiber Reinforced Plastic Composites, R.T. Schwartz and H.S. Schwartz, Eds. Interscience, (1968).

3. D.F. Adams, D.R. Doner and R.L. Thomas - Mechanical Behavior of Fiber-Reinforced Composite Materials - AFML-TR-67-96, (1968).

4. P. Hill - Theory of Mechanical Properties of Fiber-strengthened Materials - J. Mech. Phys. Solids, 12, 213, (1964).

5. Z. Hashin - Theory of Fiber Reinforced Materials - NASA CR-1974, (1972).

6. P.H. Petit and M.E. Waddoups - A Method of Predicting for the Nonlinear Behavior of Laminated Composites - J. Composite Mats. 3, 2, (1969).

7. D.F. Adams - Inelastic Analysis of a Unidirectional Composite Subjected to Transverse Normal Loading - J. Composite Mats. 4, 310, (1970).

8. W. Huang - Plastic Behavior of Some Composite Materials - J. Composite Mats. 5, 320, (1971).

9. R.L. Foye and D.J. Baker - Design/Analysis Methods for Advanced Composite Structures - AFML-TR-70-299, Vol. I, (1971).

10. B. Budiansky - A Reassessment of Deformation Theories of Plasticity - J. Appl. Mech., 26, 259, (1959).

11. W. Ramberg and W.B. Osgood - Description of Stress-Strain Curves by Three Parameters - NASA TN 902, (1943).

12. A.A. Iliouchine - Plasticite - ed. Eyrolles, (1956).

13. L.D. Wall, Jr. and M.F. Card - Torsional Shear Strength of Filament-Wound Glass-Epoxy Tubes - NASA TN D-6140, (1971).

14. R.L. Foye - Theoretical Post-Yielding Behavior of Composite Laminates - J. Composite Mats. 7, 178, (1973).

15. J. F. Mulhern, T.G. Rogers and A.J.M. Spencer - A Continuum Theory of a Plastic-Elastic Fibre-Reinforced Material - Int. J. Engng. Sci., 7, 129, (1969).

16. D.C. Drucker - Variational Principles in the Mathematical Theory of Plasticity - Symposium in Applied Mathematics, Vol. 6, 7, McGraw-Hill, (1956).

17. B. Budiansky and R.J. Vidensek - Analysis of Stresses in the Plastic Range Around a Circular Hole in a Plate Subjected to Uniaxial Tension - NASA TN 3542, (1955).

18. S.W. Tsai and E.M. Wu - A General Theory of Strength for Anisotropic Materials - J. Comp. Mats. 5, 58, (1971).

19. O. Hoffman - The Brittle Strength of Orthotropic Materials - J. Comp. Mats. 1, 200, (1967).
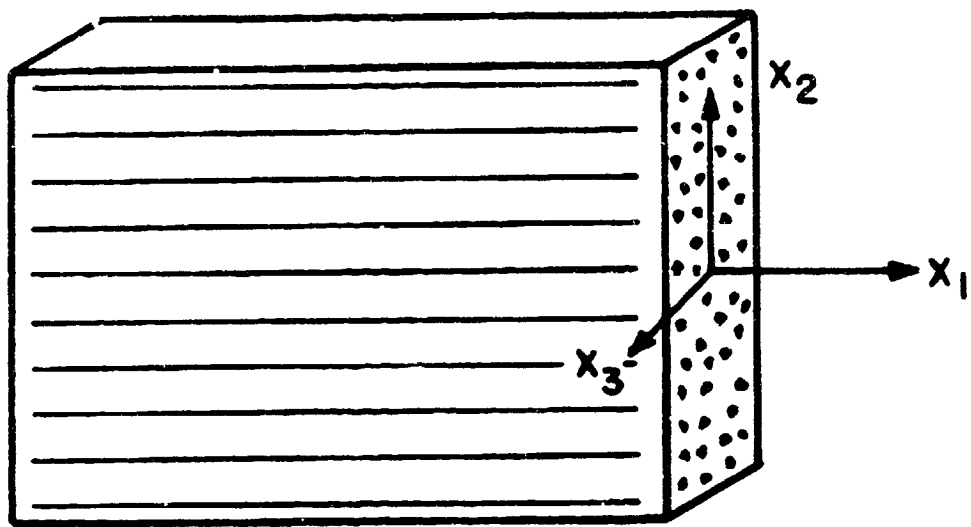
Fig. 1 -   Coordinate system for unidirectional fiber composite
           material.

74.
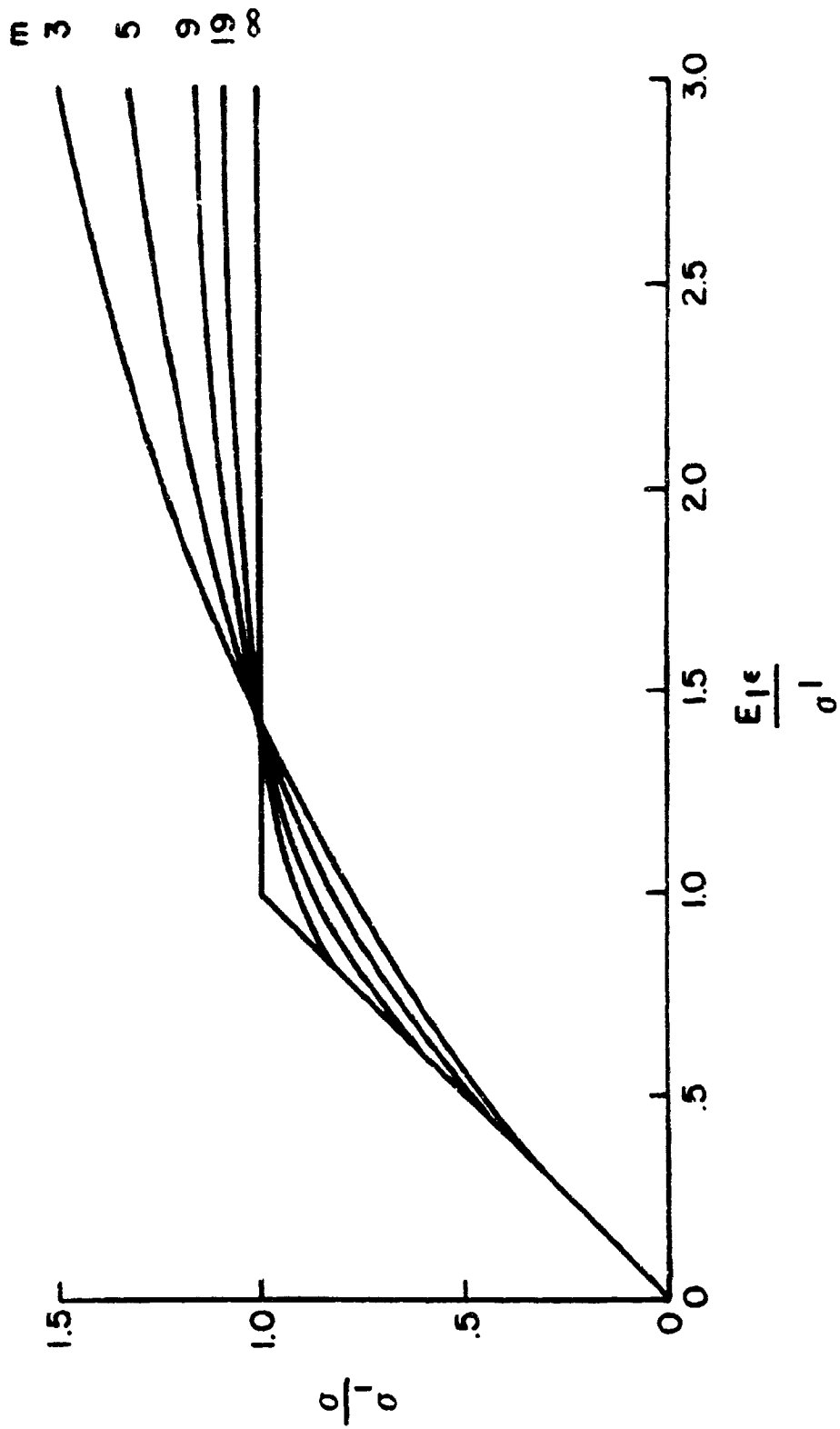
Fig. 2 - Nondimensional Ramberg-Osgood stress-strain curves. (e.g., 2.2.1 with k = 0.4).
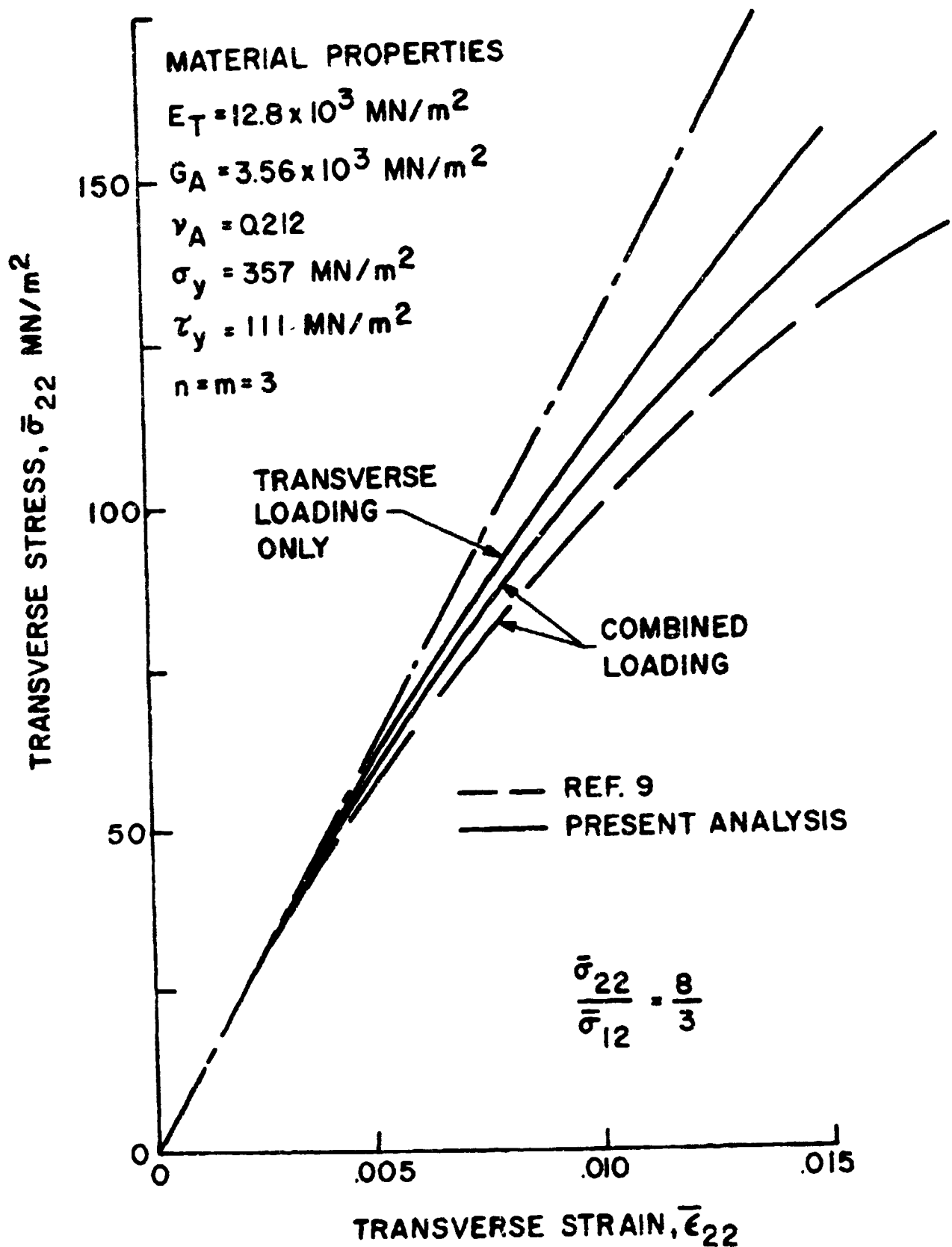
Fig. 3 - Transverse normal σ-ε curves for unidirectional Boron/
Epoxy (for normal loading and normal + shear loading).
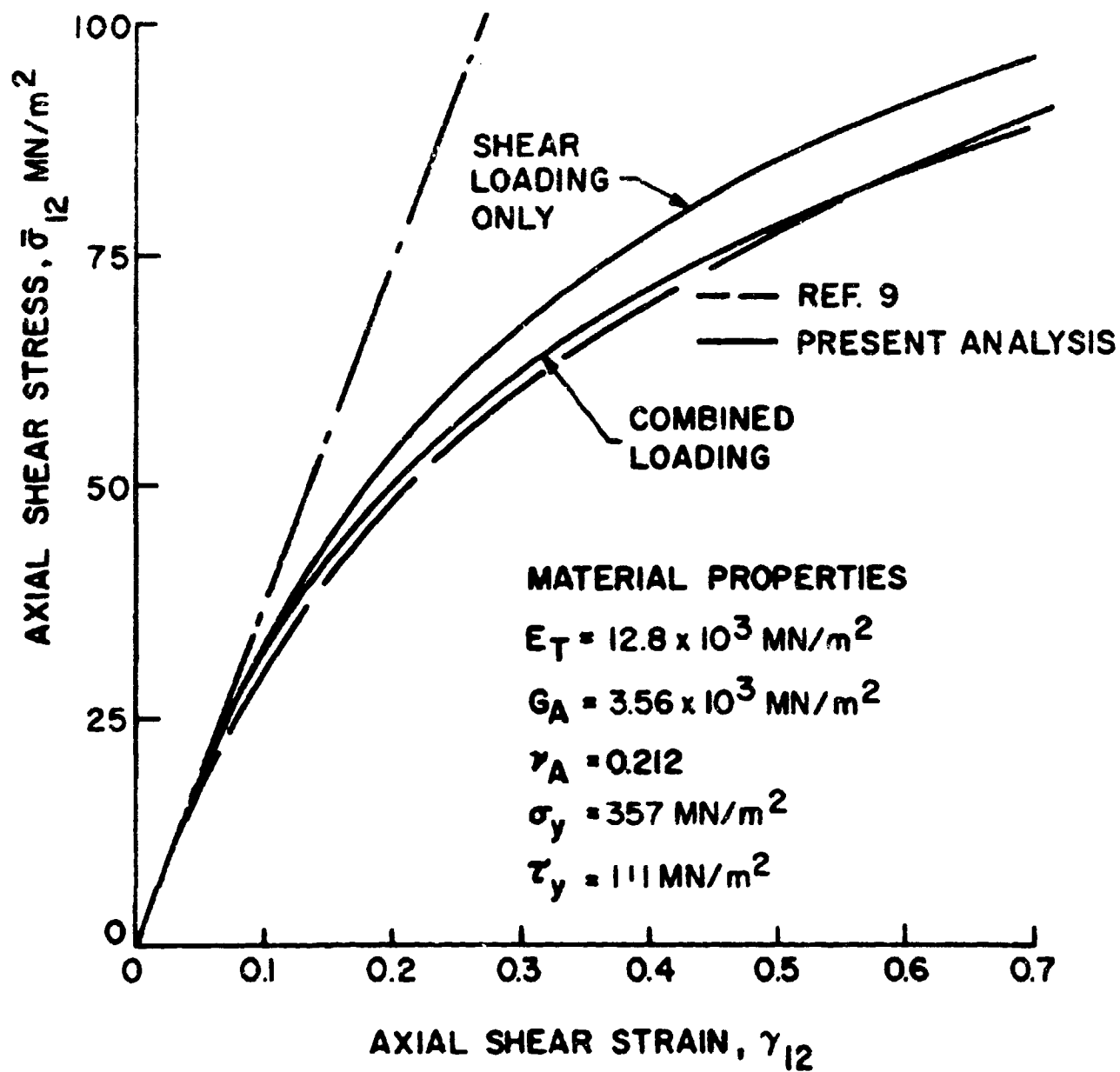
Fig. 4 - Shear σ-ε curves for unidirectional Boron/Epoxy (for shear loading and normal + shear loading).

77.
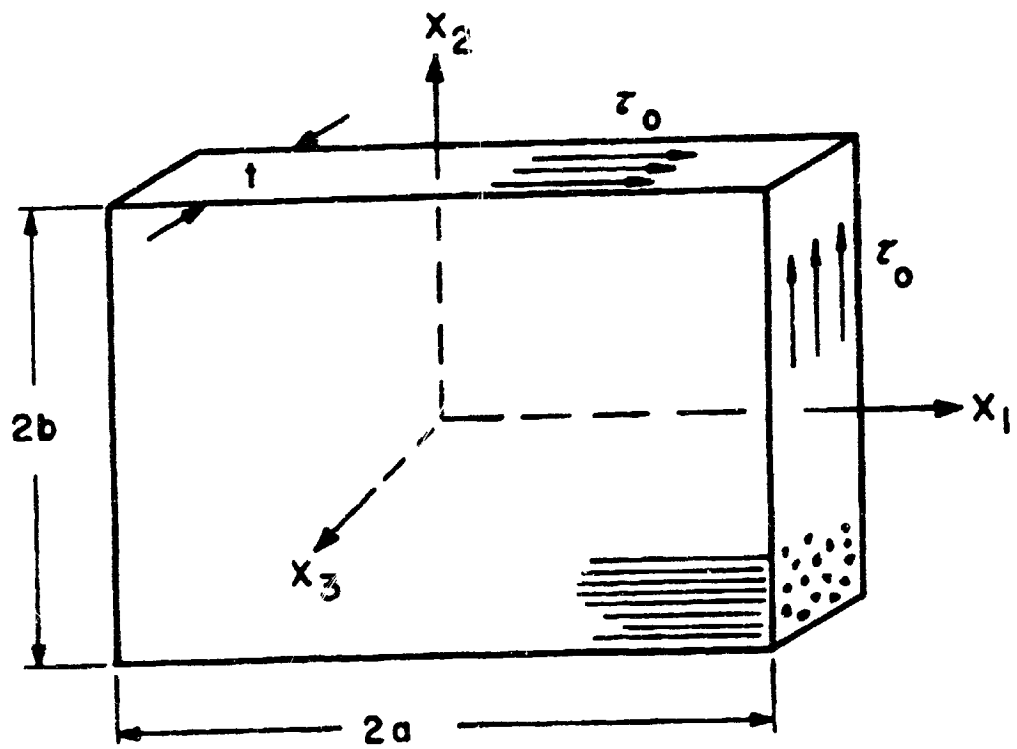
Fig. 5 - Unidirectional fiber composite material under axial
         shear stress.

78.
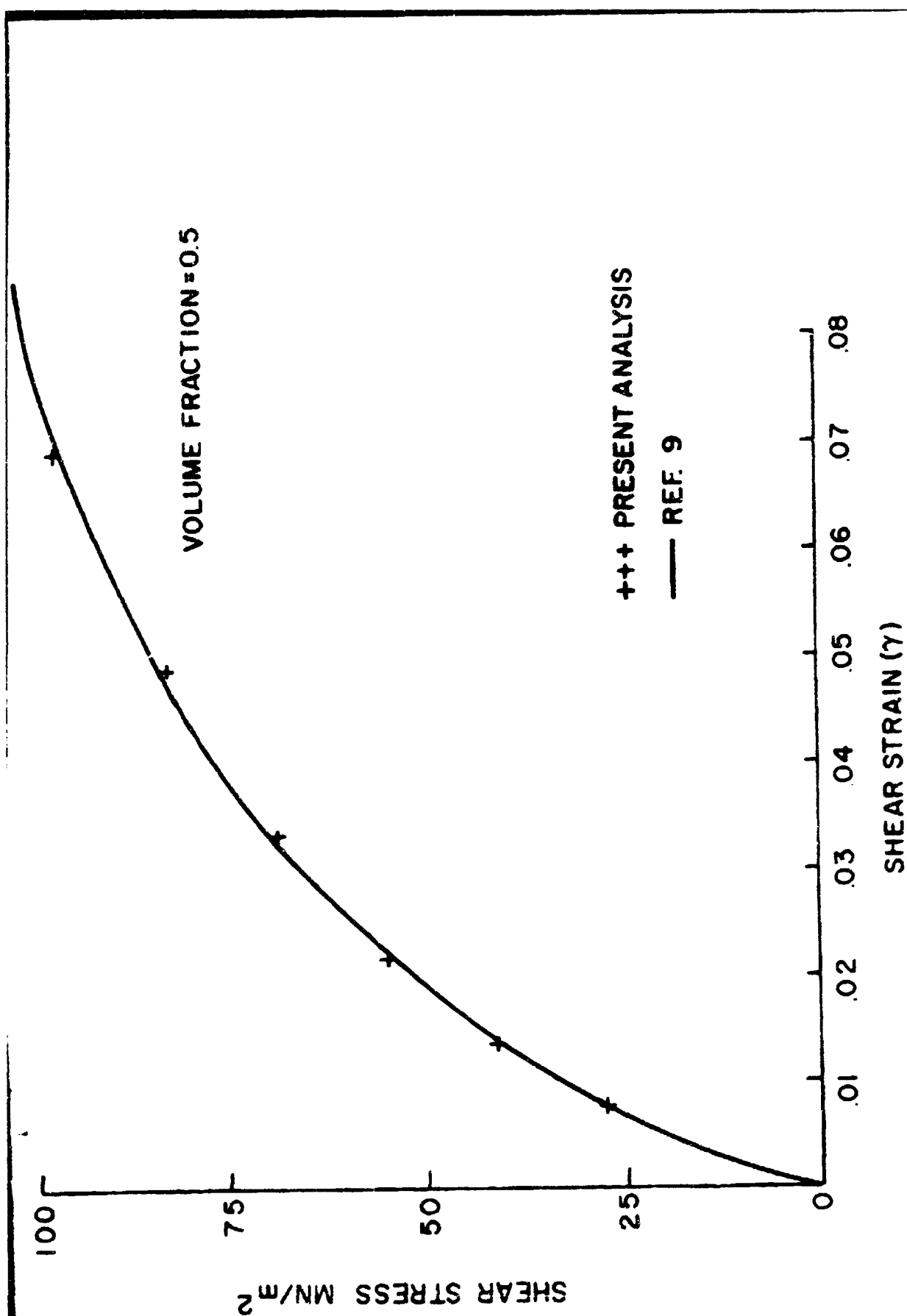
Fig. 6 - Axial shear stress-strain curve for composite computed from matrix properties.
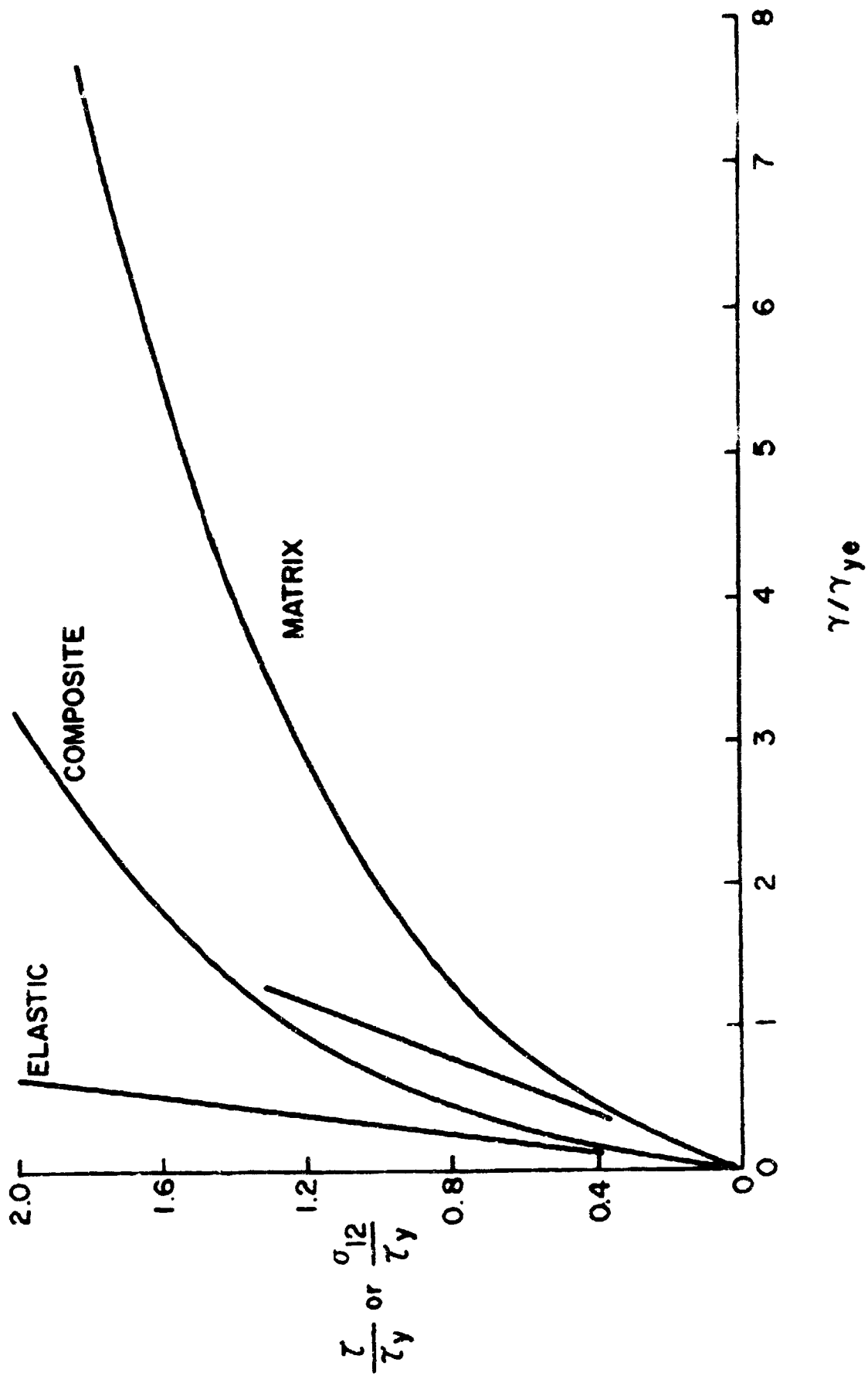
Fig. 7 – Generalized Ramberg-Osgood stress-strain curve in axial shear for composite and matrix (n=3).
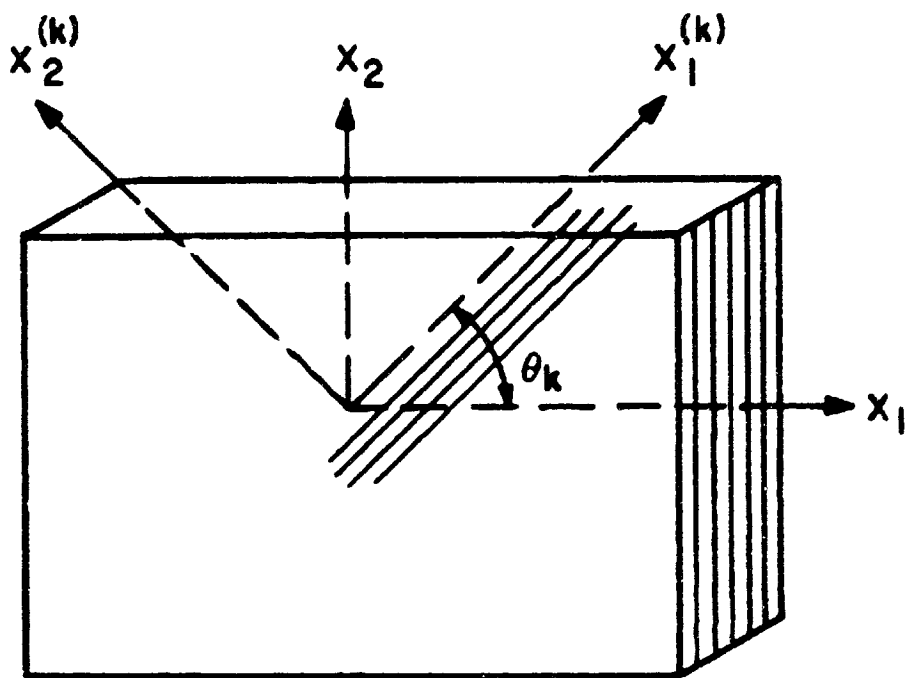
Fig. 8 - Laminate coordinate system.
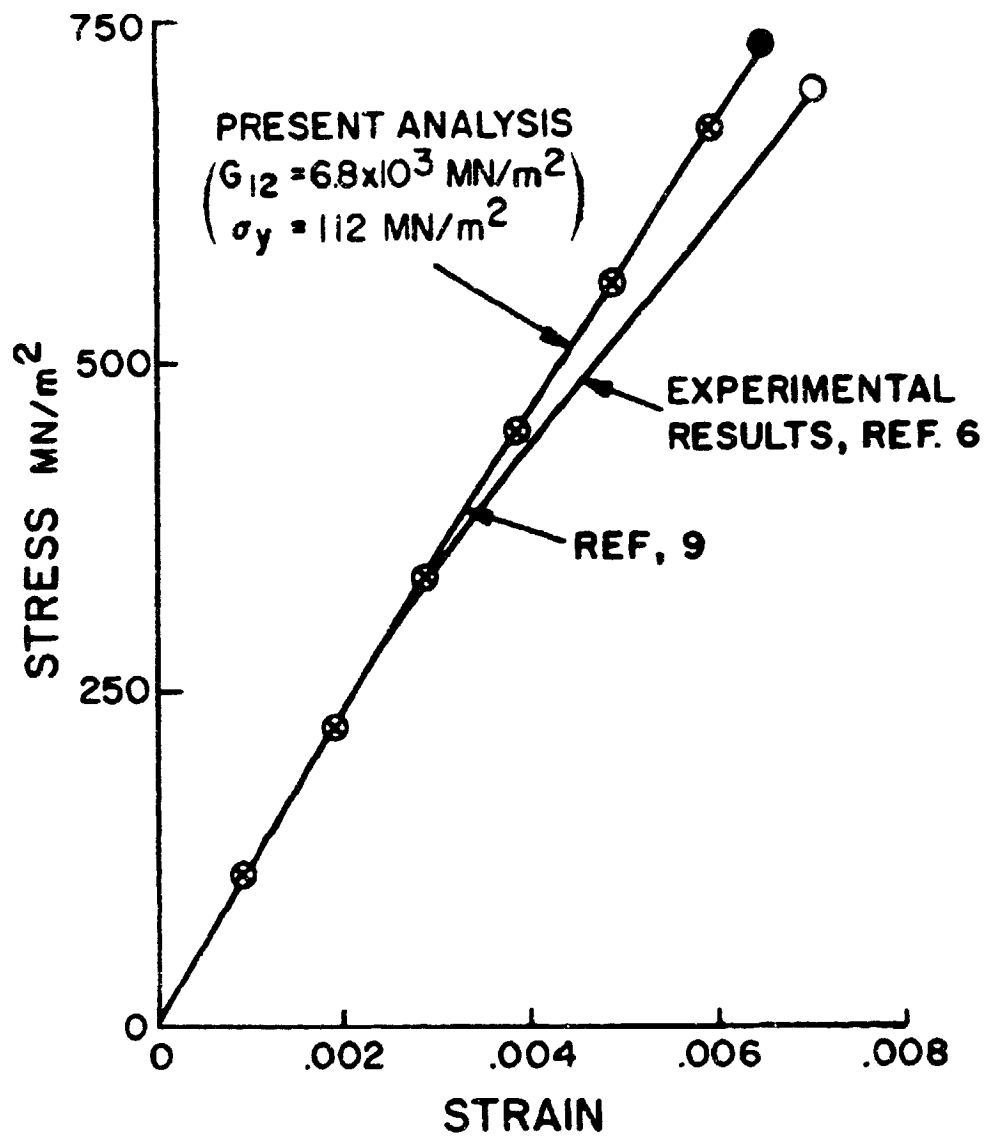
Fig. 9 - Tensile stress-strain curve with [0/90] B/Ep.
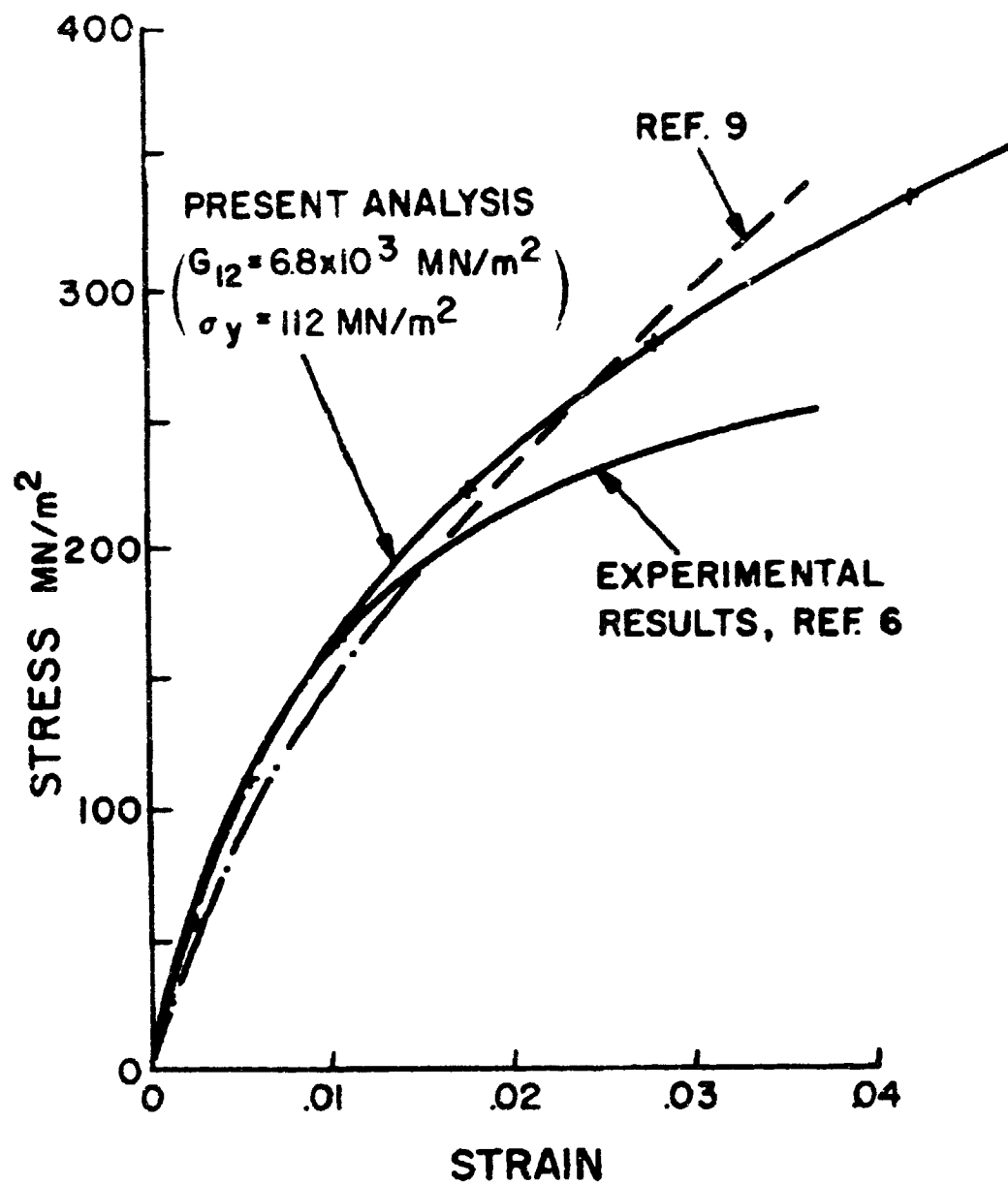
Fig. 10 - Tensile stress-strain curves for [±45] B/Ep.
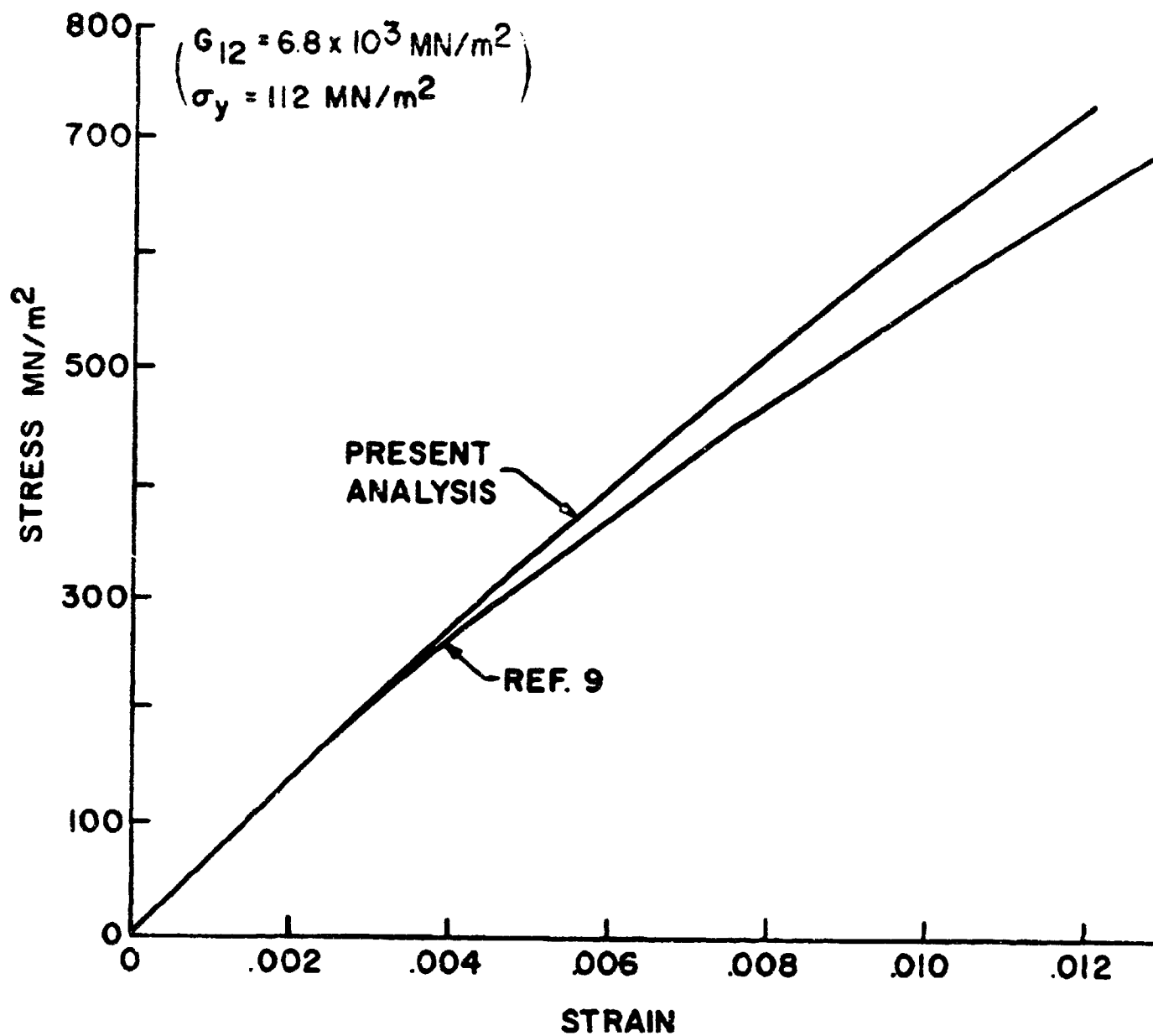
Fig. 11 - Tensile stress-strain curve with [±30] B/Ep.

84.
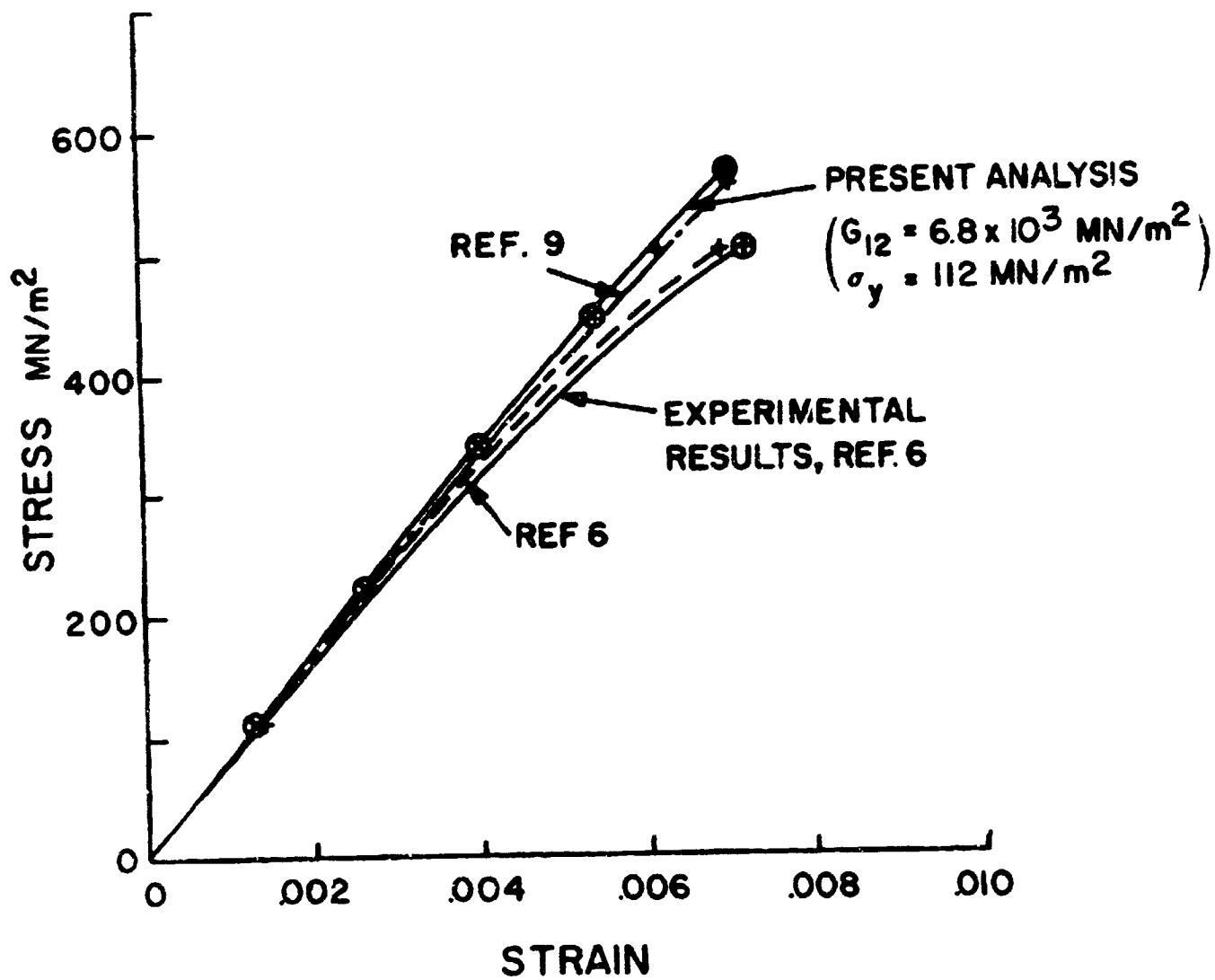
Fig. 12 - 0° Tensile stress-strain curves for {0/±45/90} B/Ep.

PRESENT ANALYSIS
$\left(\begin{array}{l} G_{12} = 6.8 \times 10^3 \text{ MN/m}^2 \\ \sigma_y = 112 \text{ MN/m}^2 \end{array} \right)$

REF. 9

EXPERIMENTAL
RESULTS, REF. 6

REF. 6

STRESS MN/m^2
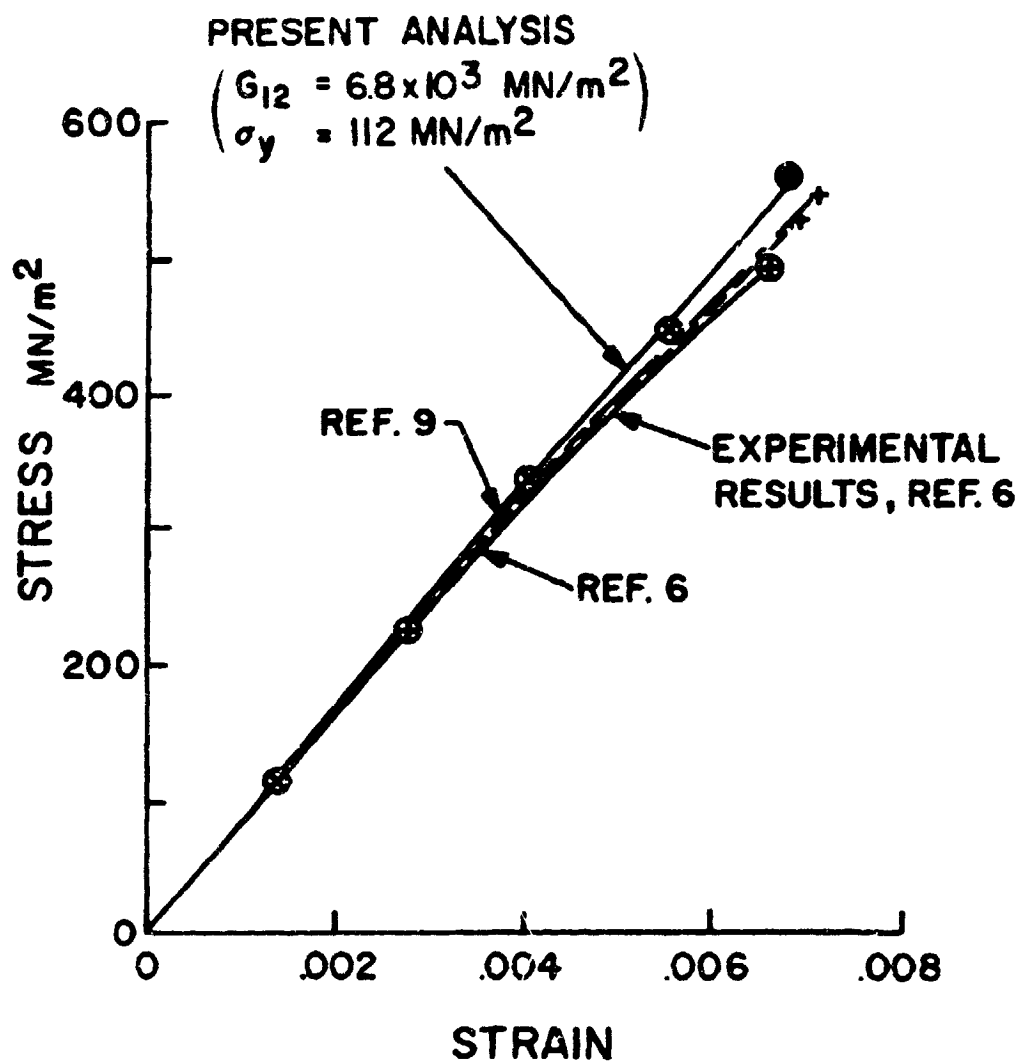
STRAIN

Fig. 13 - 0° Tensile stress-strain curves for [0/±60] B/Ep.
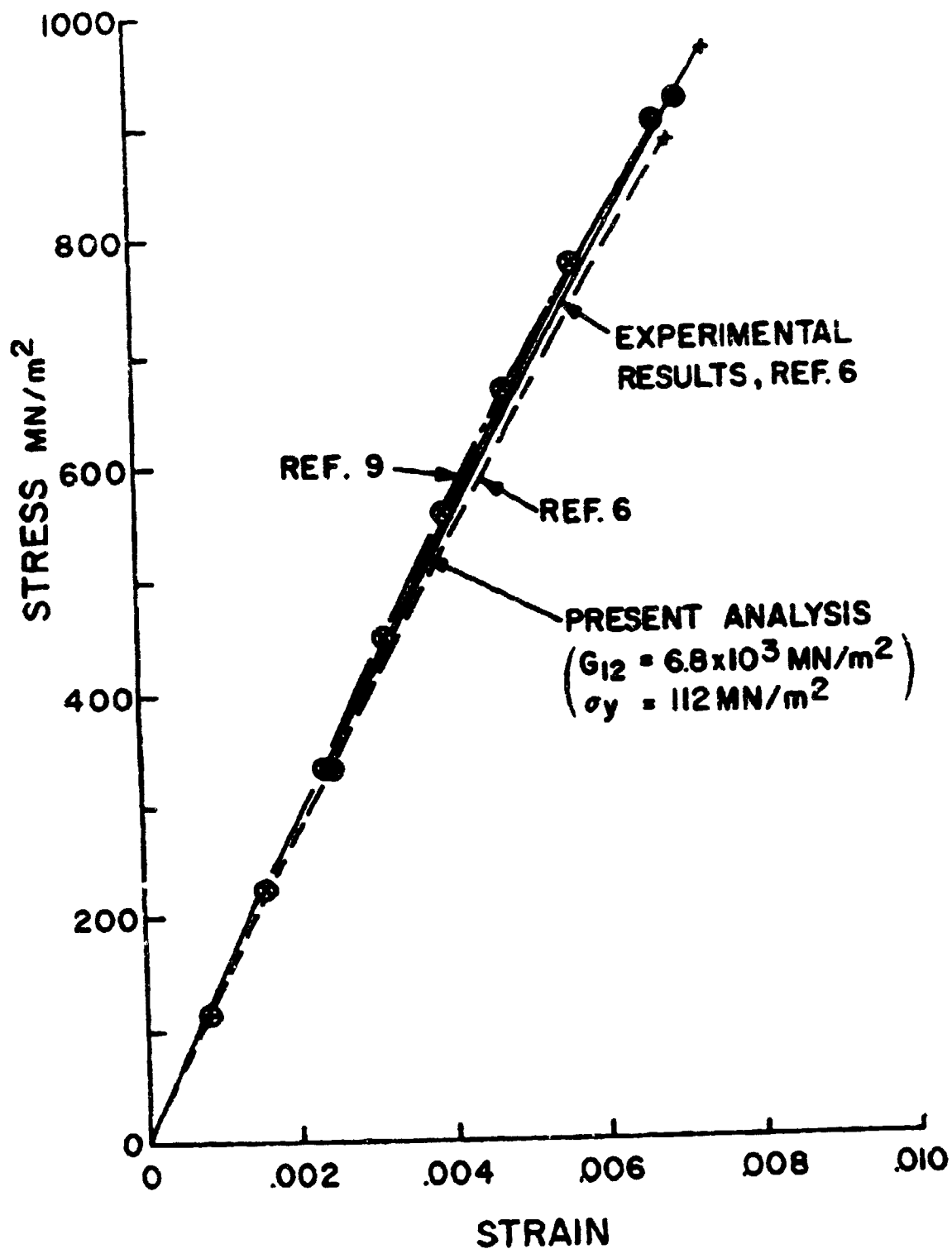
Fig. 14 - 0° Tensile stress-strain curves for [0₃/±45] B/Ep.   87.
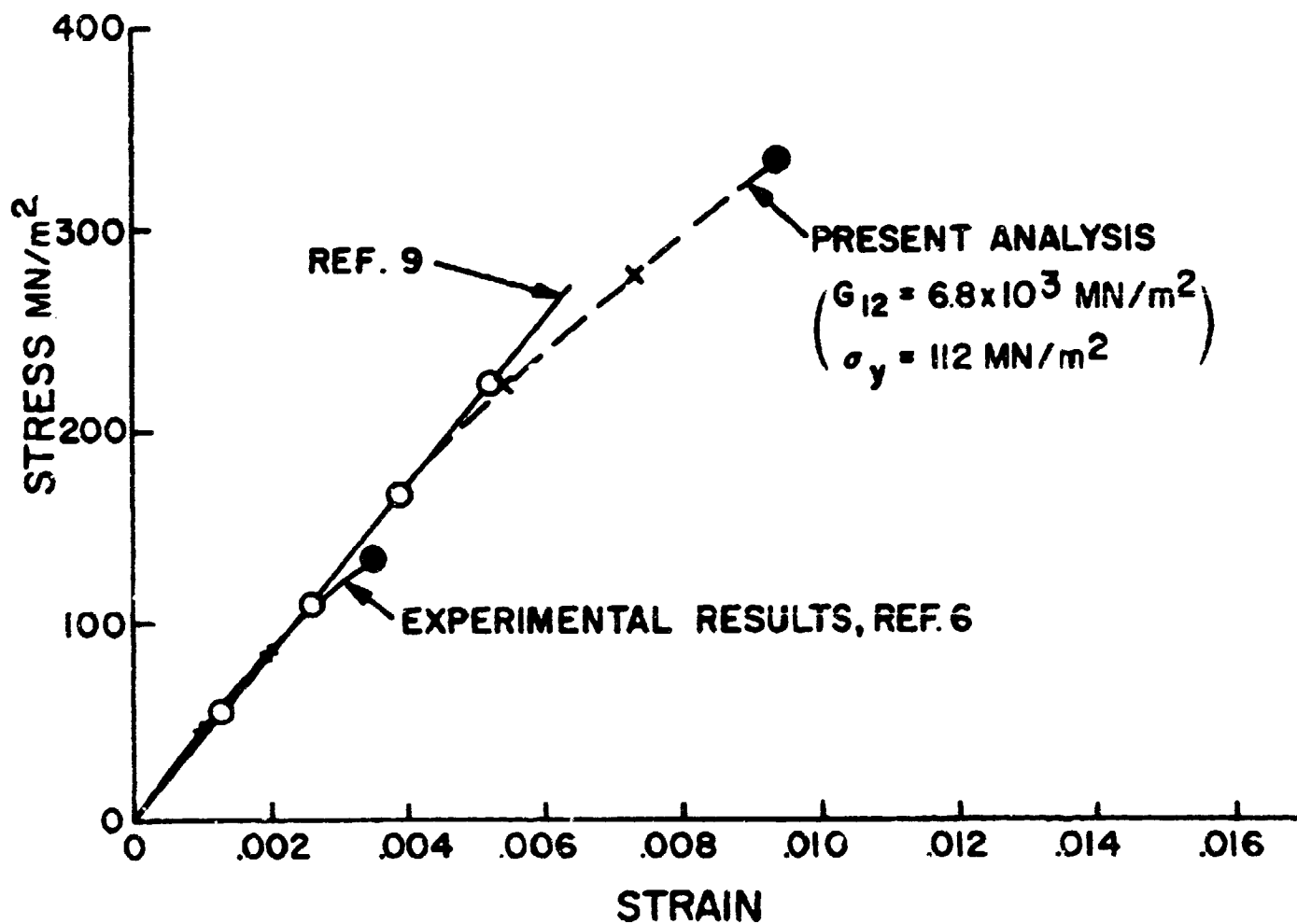
Fig. 15 - 0°Tensile stress-strain behavior of $[65_3/20/-70]$ B/Ep.

88.
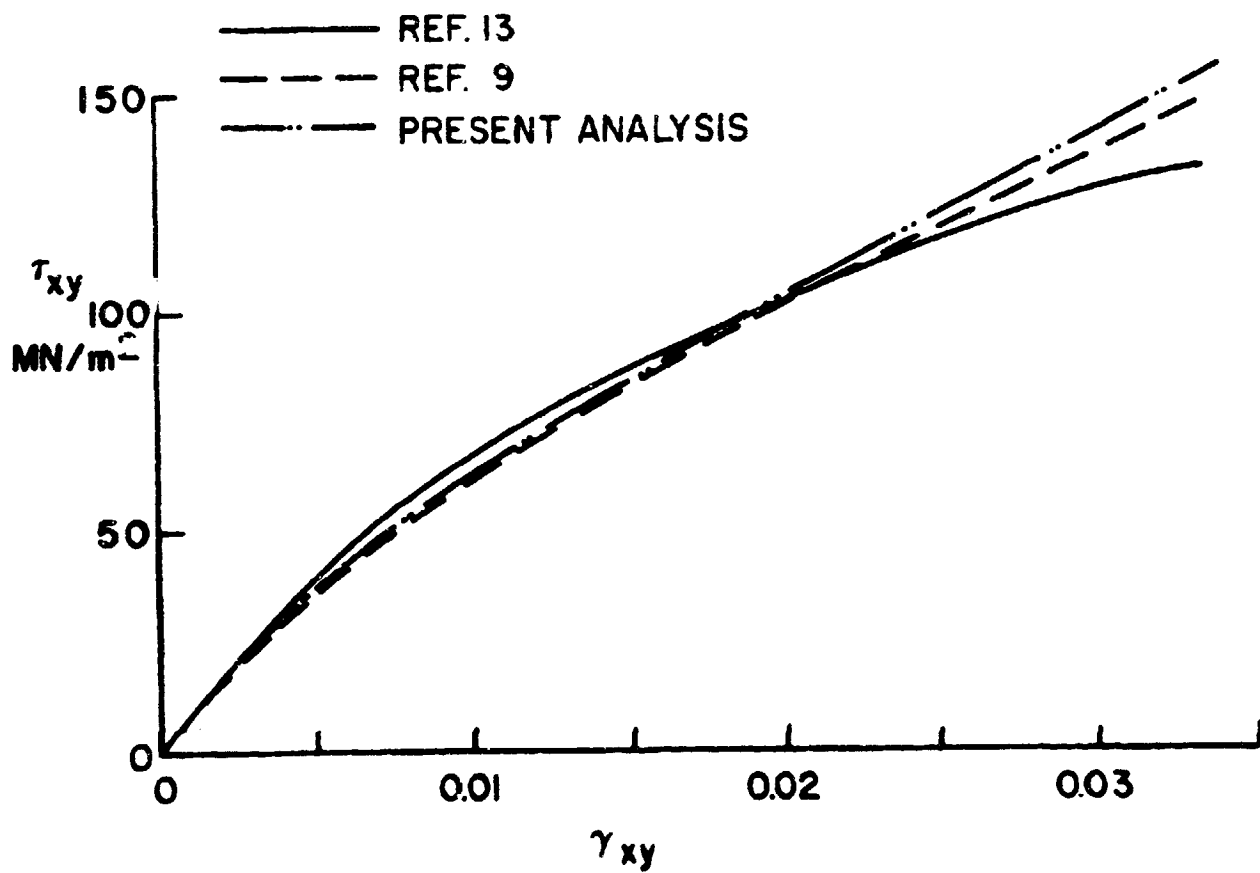
Fig. 16 - Comparison of present results with experimental data for Glass/Epoxy 90/±18 tubes in torsion, and with the results of Ref. 9.

89.
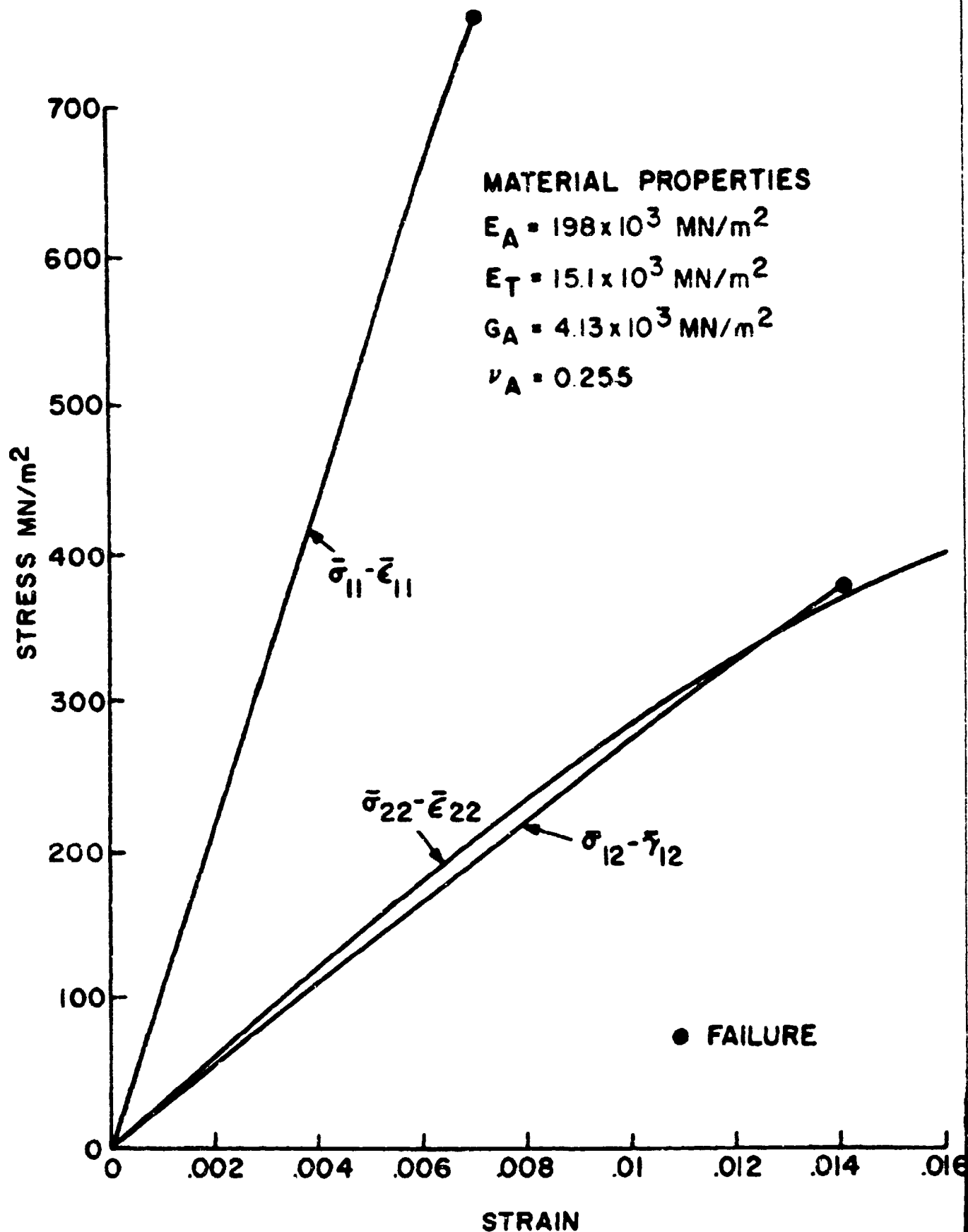
Fig. 17 - Stress-Strain curves for Axial, transverse and shear loading for [0°/±45°] laminate.
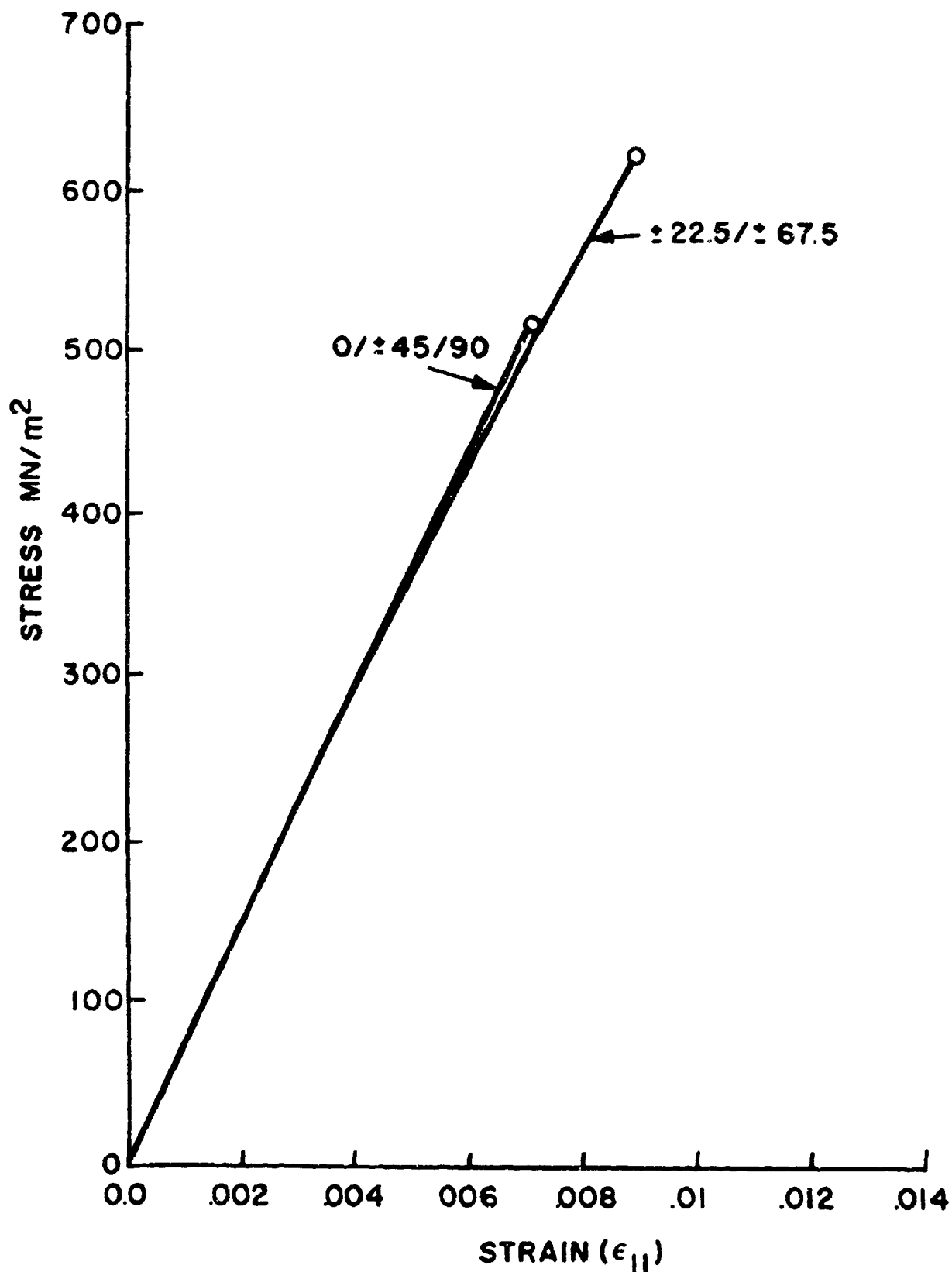
Fig. 18 – Four directional quasi-isotropic Boron/Epoxy plate under unidirectional tension in fiber direction and between fiber directions.

91
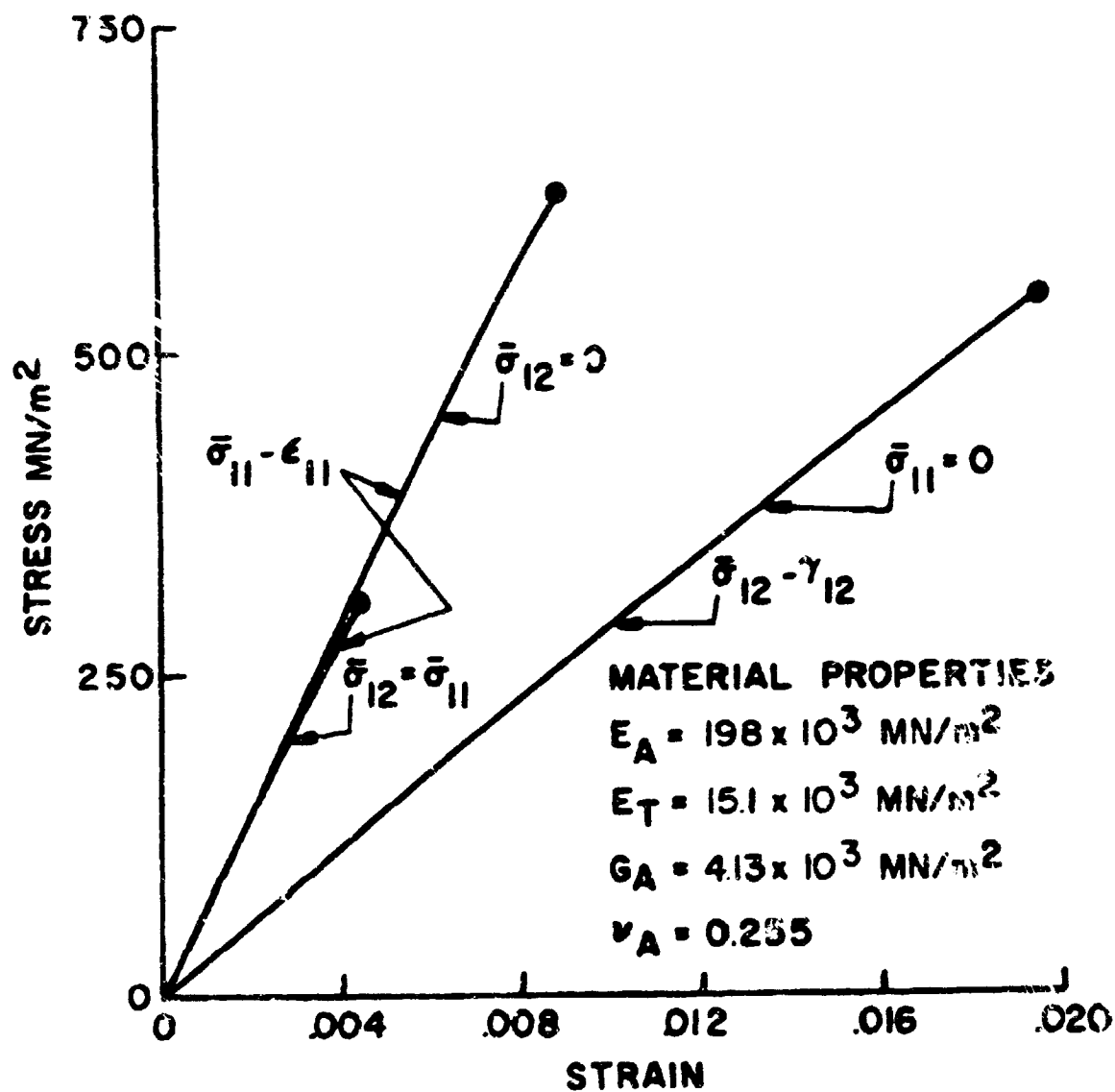
Fig. 19 - Combined stress effects on quasi-isotropic [±22.5/±67.5] laminate
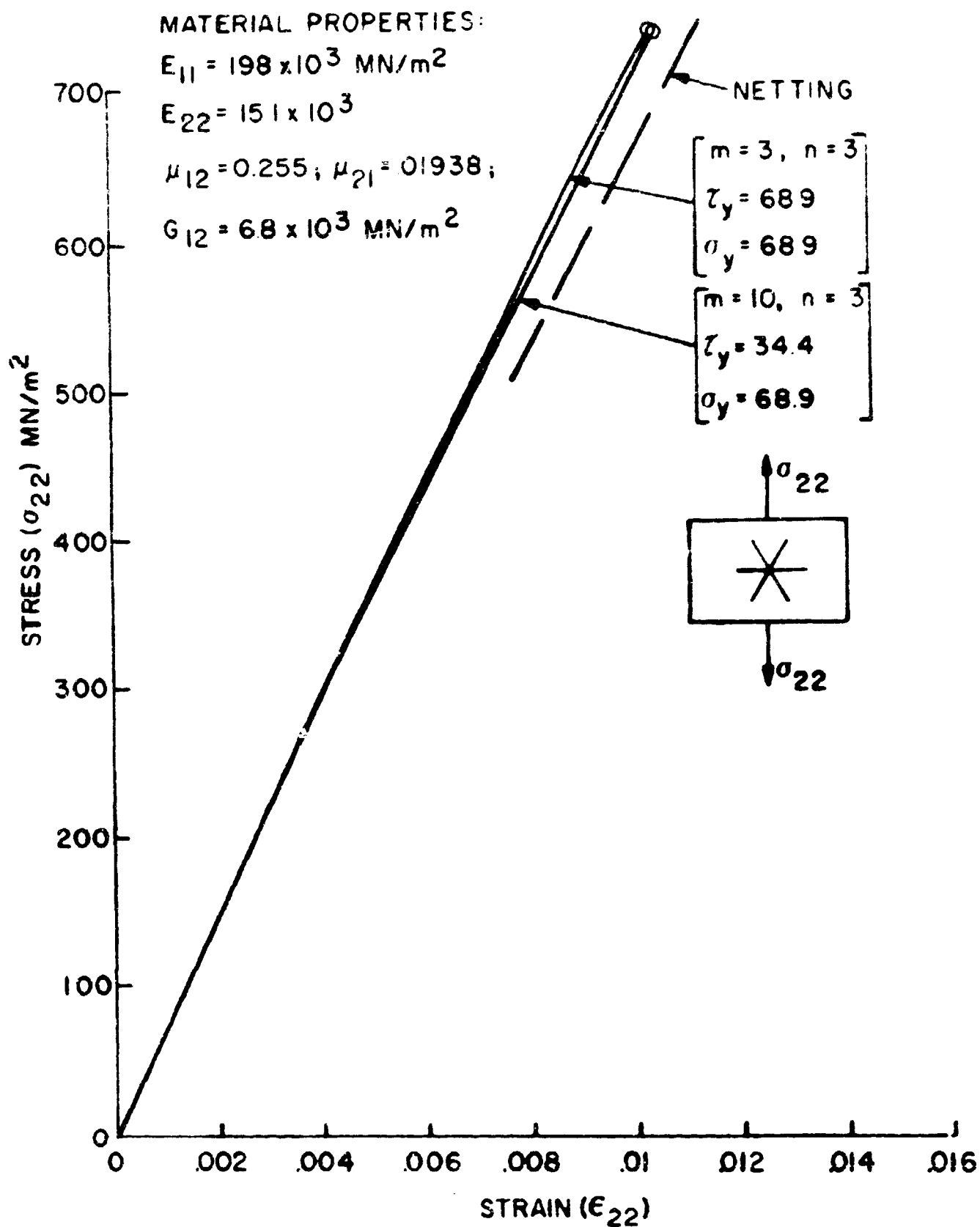
Fig. 20 - Effect of laminae inelasticity on transverse stress-strain curve for 0/±60 quasi-isotropic laminate.

93.
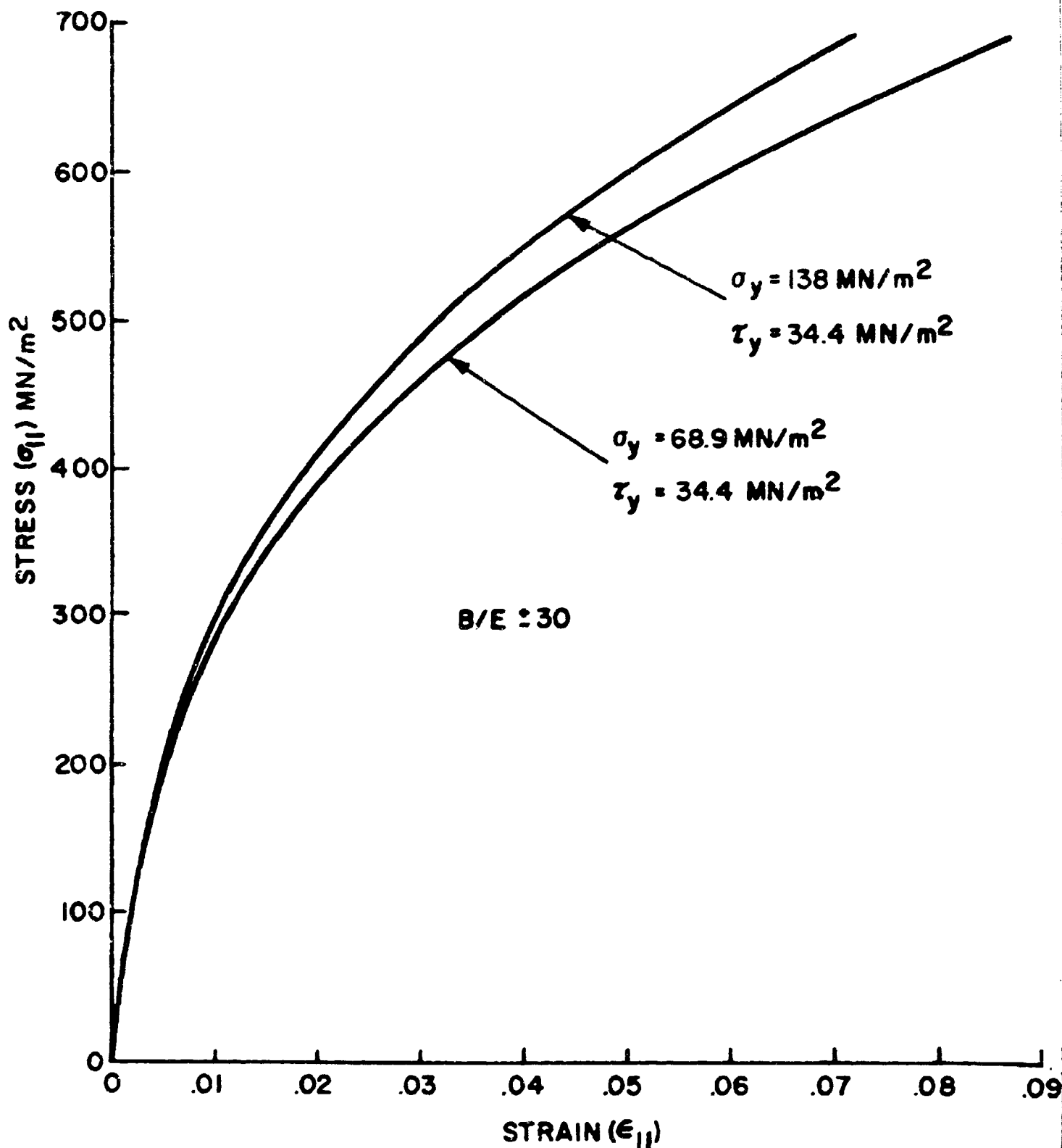
Fig. 21 - Influence of laminae inelasticity upon axial tensile
stress-strain curve of ±30° Boron/Epoxy laminate.

94.
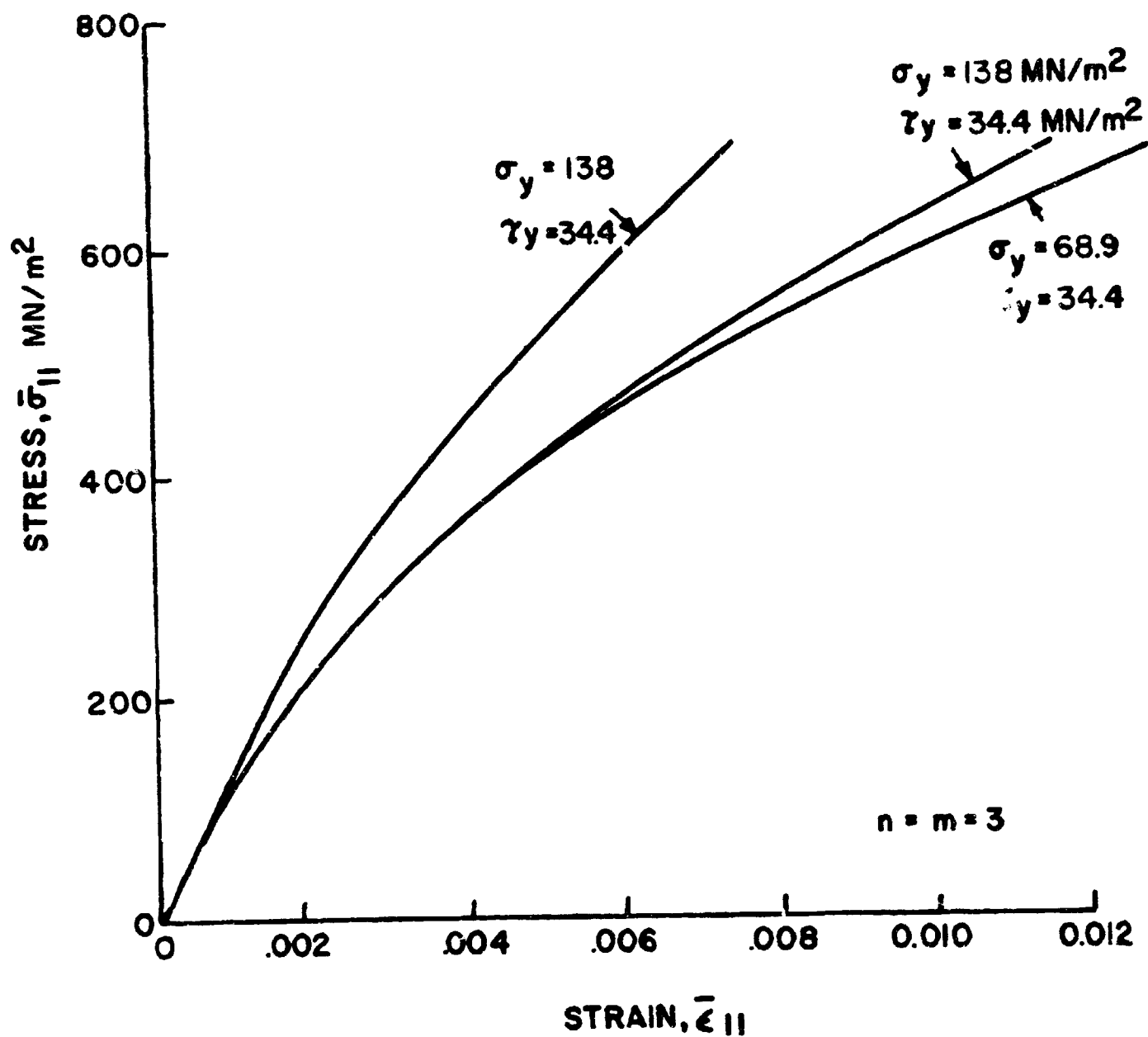
Fig. 22 - Influence of laminae inelasticity upon axial tensile
stress-strain curve of ±30° Boron/Aluminum laminate.

MATERIAL PROPERTIES

$E_A = 230.8 \times 10^3 \ MN/m^2$

$E_T = 172.2 \times 10^3 \ MN/m^2$

$\nu_A = 0.25$

$G_A = 55.9 \times 10^3 \ MN/m^2$

$\sigma_y = 68.9 \ MN/m^2$

$\tau_y = 34.4 \ MN/m^2$

$n = m = 3$

$\sigma_y = 137.8, \ \tau_y = 68.9$

$\sigma_y = 137.8, \ \tau_y = 34.4$

$\sigma_y = 68.9, \ \tau_y = 34.4$

STRESS $\bar{\sigma}_{22}$ MN/m^2

STRAIN, $\bar{\epsilon}_{22}$

Fig. 23 - 0° Tensile stress-strain curves for $[0_3/\pm45]$ B/Ep.

96.

MATERIAL PROPERTIES

$E_A = 230.8 \times 10^3$ MN/m^2

$E_T = 172.2 \times 10^3$ MN/m^2

$\nu_A = 0.25$

$G_A = 55.9 \times 10^3$ MN/m^2

$\sigma_y = 68.9$ MN/m^2

$\tau_y = 34.4$ MN/m^2

$n = m = 3$

$\dfrac{\bar{\sigma}_{12}}{\bar{\sigma}_{11}} = 0.0$

0.5

1.0

2.0

STRESS $\bar{\sigma}_{11}$ MN/m^2
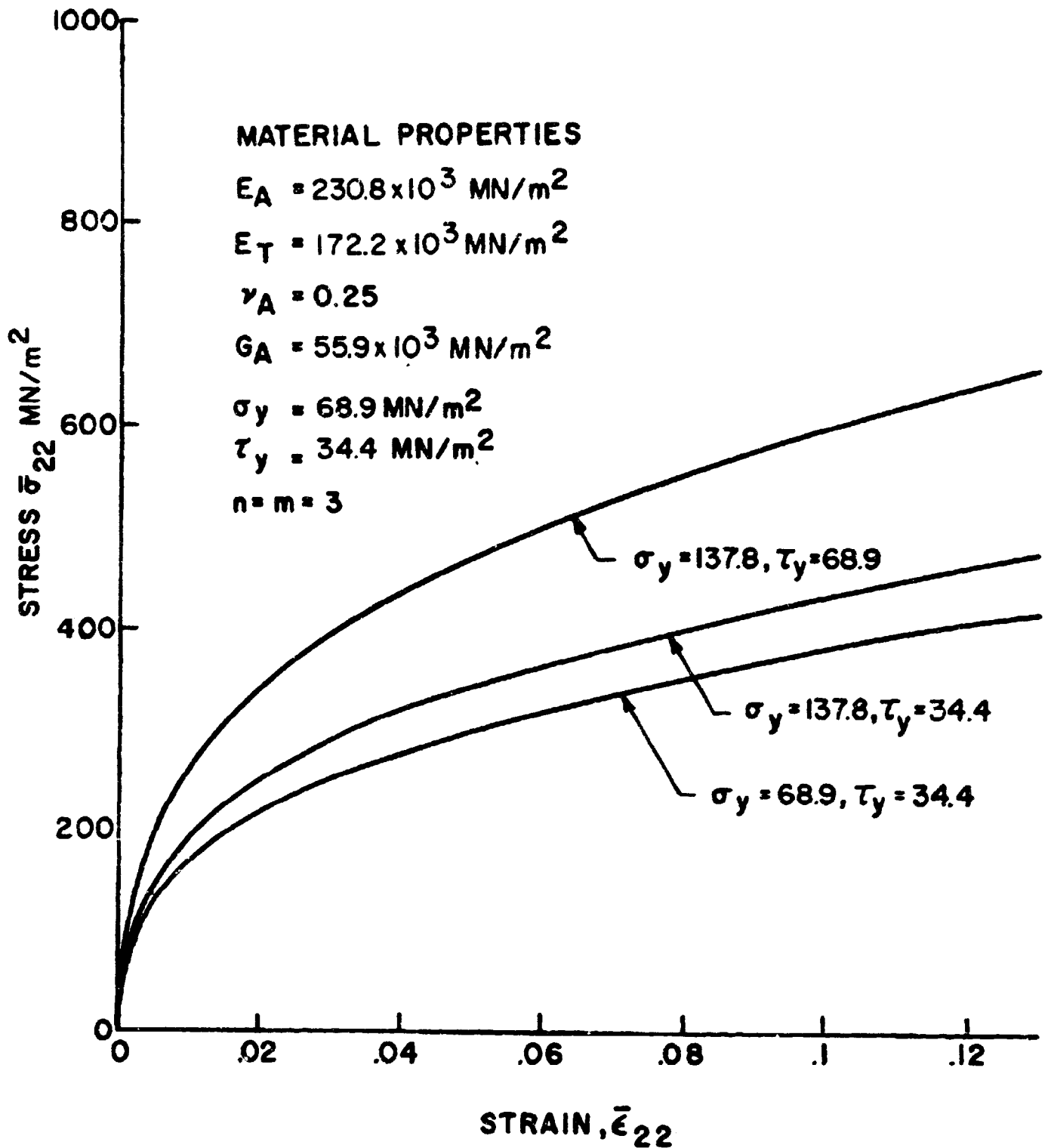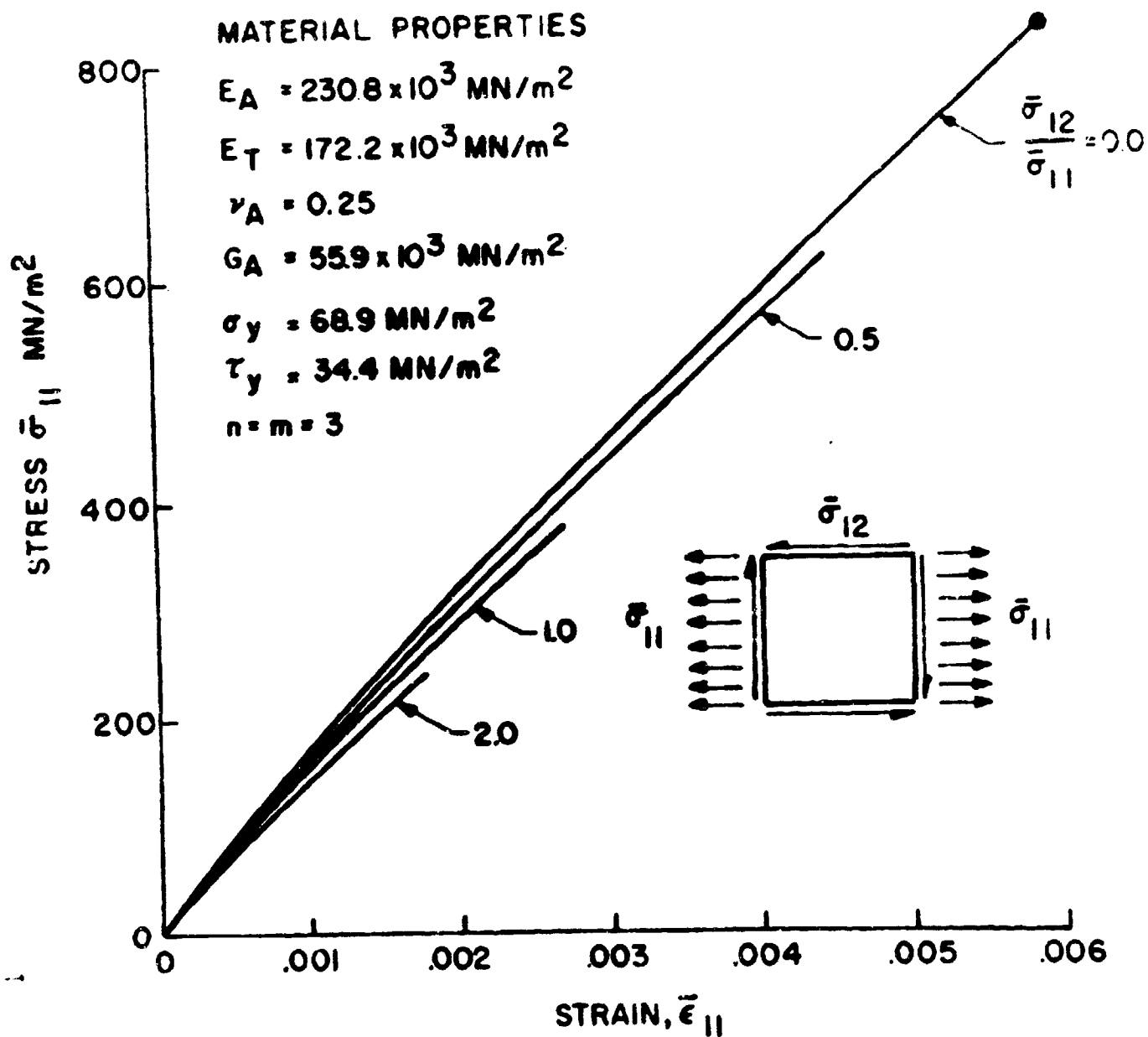
STRAIN, $\bar{\epsilon}_{11}$

Fig. 24 - Stress-strain curves of Boron/Aluminum [0/±30] laminate under combined loading.
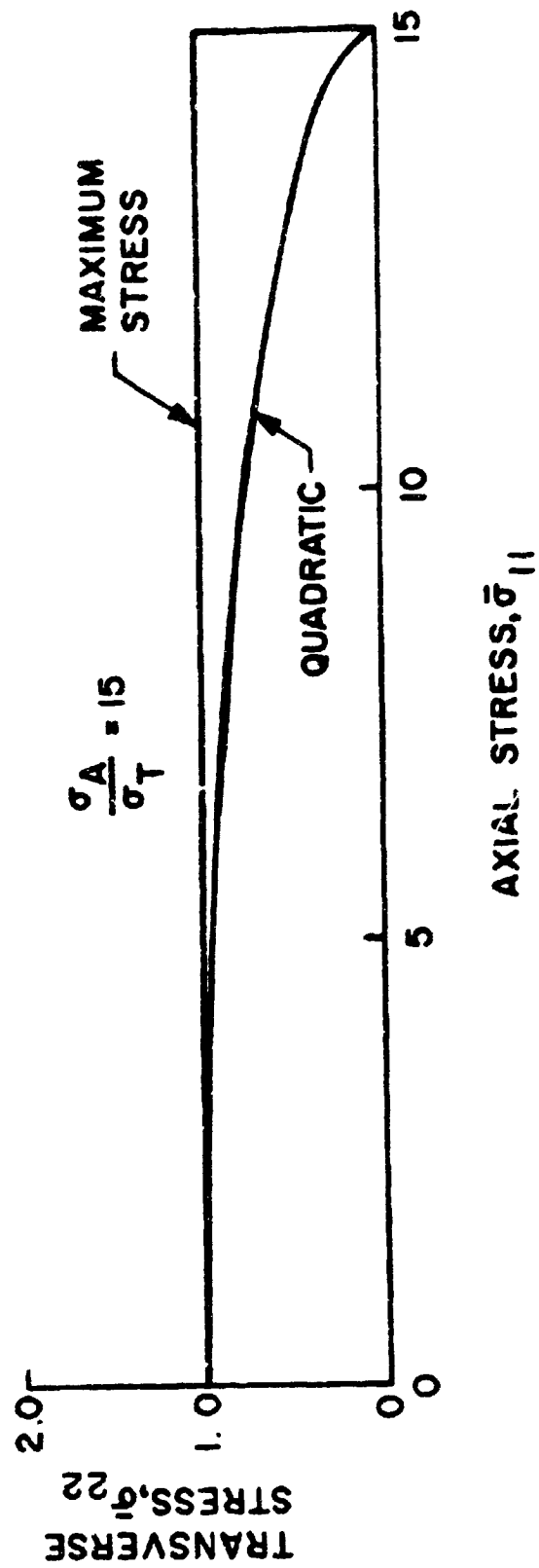
Fig. 25 - Comparison of failure criteria.

98.
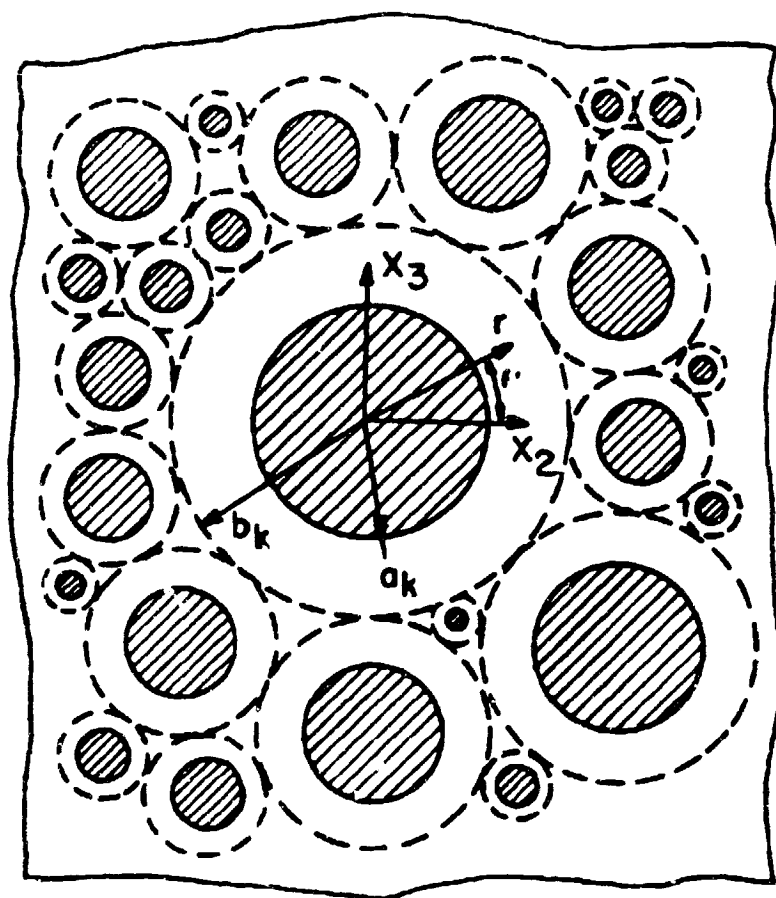
Fig. 26 - Composite Cylinder Assemblage

```
C     INELASTIC LAMINATE ANALYSIS
C
C
C
0001        IMPLICIT REAL*8(A-H,O-Z)
0002        DIMENSION E11(20),E22(20),V12(20),V21(20),G12(20),SX1(20),
     1      S11(20),S12(20),SGC(20,1),S22(20),S44(20),TANG(20),
     2      A(20,20),SGC(20,1),SGC(20,1),SF1(20),TANG(20),
0003        DIMENSION SINS(20),CSS(20),SIN2(20),CS2(20),
     1      P11(20),P22(20),P12(20),PS1(20),PS2(20),
     2      FS1(20),F11(20),FP22(20),FP12(20),FPS1(20),
     3      PS22(20),PS12(20),SGS(20),DIF(20),SG2(20),TY(20),
     4      T(3,20),DIS(3,20),HS(3,20)
0004        COMMON /ANGD/SIN,COS,SIN,COSS
0005        COMMON /CCPARM/EPS,DUBU
0006        COMMON /IN.PARM/11,NIT
0007        COMMON /TLPRM/SGP,STIFF
          COMMON /ENTRM/EP11,EPS11,SM1,SC11
0008        LOGICAL MSING,MSINGG
C
C
C     INPUT
C
C     NOTE: CONVENTION FOR V12 AND V21 ESTABLISHED BY FOLLOWING
C          RELATIONSHIPS-          S12 = -V12/E11
C                                  S21 = -V21/E22
C
0009        WRITE(6,1500)
0010        READ(5,10)C) NSETS
0011        DO 999 J = 1,NSETS
0012        READ(5,1010) LAY
0013        READ(5,1030) INP
C
C     LAMINATE OUTPUT HEADING
0014        WRITE(6,1509) J
0015        WRITE(6,1481)
0016        WRITE(6,1210) LAY
C
0017        GO TO (20,25), INP
C
C     INPUT 1:
0018   20   CONTINUE
0019        DO 24 I = 1,LAY
0020        READ(5,1002) E11(I),E22(I),V12(I),V21(I)
0021        READ(5,1002) G12(I),SX(I),TY(I)
0022        READ(5,1022) TTI,TANG(I)
0023   24   CONTINUE
          GO TO 50
C
C     INPUT 2:
0025   25   CONTINUE
0026        WRITE(6,1561)
0027        WRITE(6,1500)
0028        DO 29 I = 1,LAY
0029        CALL INPUT(TTI,E22,V12,V21,G1,SY,TY,I)
0030        READ(5,1022) TTI,TANG(I)
0031
```

```
0032          29 CONTINUE
                 C
0033          50 CONTINUE
                 C
0034          C        EQUATION PARAMETERS
                       READ(5,1002) XN,XM
                 C     INITIAL LOADING
0035                   READ(5,1002) SQ11,SQ22,SQ12
                 C     INCREMENTATION PARAMETER
0036                   READ(5,1024) KSU,SMI
                 C     LAMINATE TEST PARAMETERS
0037                   READ(5,1002) SILF
0038                   READ(5,1002) SGR
                 C     CONTROL PARAMETERS
0039                   READ(5,1024) IT,EPS,UPHD
0040                   READ(5,1013) INMT
                 C
0041             110 1=1
0042                   WRITE(6,1513)
0043                   DO 60 I=1,LAY
                       WRITE(6,1515)   I,TANG(I),T(I),E11(I),E22(I),V12(I),V21(I),
                                       G12(I),SV(I),TY(I)
                 C
0044          60 CONTINUE
0045                   WRITE(6,1516)
0046                   WRITE(6,1517) XM
0047                   WRITE(6,1518) XN
0048                   WRITE(6,1520)
0049                   WRITE(6,1503)
0050                   WRITE(6,1549)
                 C
                 C     ANGLE REDUCTION ROUTINE
0051                   CALL ANGF(TLAY,TANG)
                 C
                 C     INITIAL ASSIGNMENTS AND COMPUTATIONS
0052                   TT = 0.0D0
0053                   DO 100 I = 1,LAY
0054                   TT = TT + T(I)
0055         100 CONTINUE
0056                   DO 105 I=1,70
0057                   SG(I) = 0.0000 00
0058                   SUS(I) = 0.0000
0059                   SF(I) = 1.000
0060         105 CONTINUE
0061                   SM11 = SMLT*SQ11
0062                   SM22 = SMLT*SQ22
0063                   SM12 = SMLT*SQ12
0064                   KSG = 1
                 C
0065                   LT1 = LAY
0066                   N   = LT1
0067                   LP1 = LAY +1
0068                   LT2 = LAY+2
0069                   LT21 = LAY+2-I
0070                   LT3 = LAY+3
0071                   LM1 = LAY -1
                 C
0072                   DO 107 I = 1,LAY
0073                   S11(I) = 1.0D0/E11(I)
0074                   S12(I) = -V12(I)/E11(I)
0075                   S22(I) = -V21(I)/E22(I)
0076         107 CONTINUE
                 C
```

102

```
0077        110 CONTINUE
            C
            C       RETURN TO 110 AFTER IMPLEMENTING APPLIED LOAD OF
            C       FROM LAMINATE TESTS J AND 4
            C
0078            MSING = .FALSE.
0079            MSIGID = .FALSE.
0080            NIT = 0
            C
0081            DO 111 K = 1,70
0082            DK(K) = 0.000
0083            SACK(I) = 0.000
0084            SACK,1) = 0.000
0085            SA2(K) = 0.000
0086            DO 111 L=1,70
0087            DKK,L) = 0.000
0088            AKK,L) = 0.000
0089        111 CONTINUE
0090            DO 115 I=1,N
0091            S22(I) = 1.000/E22(I)
0092            S44(I) = 1.000/(4.000/G12(I))
0093        115 CONTINUE
            C
            C       AFTER 2ND INCREMENTATION USE MULTIPLICATIVE FACTOR
            C       AS INITIAL STRESS SOLUTION ESTIMATE
0094        112 CONTINUE
0095            IF(N.GT.(I+1)) GO TO 120
0096            DO 115 I=1,IPI
            C
0097            SNS   = SINS(I)
0098            CSS   = CCSS(I)
0099            SN2   = SIN2(I)
0100            CS2   = CCS2(I)
            C
0101            IF(N.P.GT.I) GO TO 112
0102            SMSP   = SINS(I+1)
0103            CSSP   = CCSS(I+1)
0104            SN2P   = SIN2(I+1)
0105            CS2P   = CCS2(I+1)
            C
0106        112 CONTINUE
0107            A11,I     )   = CSS*T(I)
0108            A11,I+N   )   = SNS*T(I)
0109            A11,I+2*N )   = -SS*2*T(I)
            C
0110            A12,I     )   = SNS*T(I)
0111            A12,I+N   )   = CSS*T(I)
0112            A12,I+2*N )   = SN2*T(I)
            C
0113            A13,I     )   = SN2*T(I)/2.000
0114            A13,I+N   )   = SN2*T(I)/2.000
0115            A13,I+2*N )   = CS2*T(I)
0116            IF(N.GT.I) GO TO 114
            C
0117            A13*T(I,I   )   =
0118            A13*T(I,I+N )   =
0119            A13*T(I,I+2*N)  =
0120            A13*T(I,I+N*L)  =
0121            A13*T(I,I+2*N)  =   2.000*SA4(I)*SN*N*
0122            A13*T(I,I+2*N)+L= -2.000*SA4(I)+L)*SN2P
```

```
0123        A(3,1+2,1      ) = -S1(I) )*S*S -S2(I) )*CSS
0124        A(3,1+2,101    ) =  S1(I)+1)*SP*SP+S2(I)+1)*CSS
0125        A(3,1+2,10A    ) =  S1(I) ) -S2(I) )*CSS
0126        A(3,1+2,1A+1   ) =  SP/(I+1)*S*SP+S2(I)+1)*CSS
0127        A(3,1+2,1A2+N   ) =  2.0D0*S4(I) )*SP*2
0128        A(3,1+2,1A2+N+1) =  2.0D0*S4(I)+1)*S*SP
     C
0129        A(3,1+3,1      ) = -(S1(I) )-S2(I) )*SM2 /2.0D0
0130        A(3,1+3,101    ) = -(S1(I)+1)-S2(I)+1)*SC2P /2.0D0
0131        A(3,1+3,1A+N   ) = -(S1(I) )-S2(I) )*SM2 /2.0D0
0132        A(3,1+3,1A+N+1  ) = (S1(I)+1)-S2(I)+1)*SN2P/2.0D0
0133        A(3,1+3,1A2+N   ) = -2.0D0*S4(I) )*CS2
0134        A(3,1+3,1A2+N+1) =  2.0D0*S4(I)+1)*CS2P
     C
0135   113  CONTINUE
     C
     C
0136        A(1,N  ) =  CSS2*(I A)
0137        A(1,2+N) =  SNS*(I A)
0138        A(1,1+N) = -SN2P*(I N)
0139        A(2,N  ) =  SAS*(I N)
0140        A(2,2+N) =  CSS*(I N)
0141        A(2,3+N) =  SN2P*(I N)
0142        A(3,N  ) =  SN2P*(I A)/2.0D0
0143        A(3,2+N) = -SN2P*(I N)/2.000
0144        A(3,1+N) =  CS2P*(I N)
0145        IF(MSING) GO TO 117
     C
0146   116  CONTINUE
     C
     C         INVERT MATRIX A
0147        CALL INVRT(A,20,L15,0E',1.0D-15,IRANK)
     C
     C         CHECK FOR SINGULAR MATRIX
0148        IF(IRANK.EQ.L13) GO TO 118
0149        MSING = .TRUE.
0150        WRITE(6,1420) IRANK,DET
0151        GO TO 112
0152   117  CONTINUE
0153        WRITE(6,1425) ((A(I,J),I=1,L13),J=1,L13)
0154        MSING = .FALSE.
0155        GO TO 999
0156   118  CONTINUE
     C
     C         B.C. VECTOR
0157        SG(I+1) = SC11
0158        SG(2,1) =  SC22
0159        SG(4+1) = -SC12
0160        CALL MXMUL(A,SG,SG+20,SG+20,20,1,L13,L13,1)
     C
     C         RESET STRESS = 0, IF RELATIVE STRESS < 1.0D-06
0161        CALL RESET(I13,SG+1,SG-1.0D-06)
0162        GO TO 162
     C
     C         (MULTIPLICATIVE FACTOR) X (SOLUTION FROM PREVIOUS INCREMENT)
0163   120  CONTINUE
0164        DO 122 I=1,L13
0165        SG(I,1) =   SF(I)*SG(I)
0166   122  CONTINUE
0167        GO TO 126
     C
```

```
       C         RETURN TO 125 FOR NEXT ITERATION STEP
0168   125 CONTINUE
0169       GIT = NLF,A
0170   126 CONTINUE
0171       DO 127 I=1,LL13
0172       SG(I,1) = SG(I,1)
0173       DTI1,1 = 0.0D 00
0174   127 CONTINUE

0175   130 CONTINUE
0176       DO 130 K=1,LL13
0177       DO 130 I=1,LL3
0178       DB(R,I) = 0.0D 00
0179  130J CONTINUE
0180       IF(N.EC.1) GM1=1
0181       DO 151 I=1,LM
0182       CALL NRTPM (LAY,SG,...,SY,TY,AN,XM,1)
0183       CALL NRTPM (LAY,ISG,GM,GC,SY,TY,XM,XM,1,1)

0184       SGS = S*GSS(I)
0185       CSS = CGSS(I)
0186       SN2 = SIN2(I)
0187       CS2 = COS2(I)

0188       IF(N.EQ.0-1) GG TG 131
0189       SNSP = SINS(I,1)
0190       CSSP = CUSS(I,1)
0191       SN2P = SIN2(I,1)
0192       CS2P = COS2(I,1)

0193   131 CONTINUE
0194       DB(1,1)          = CSS*TI(1)
0195       DB(2,1)          = SNS*TI(1)
0196       DB1L(1+2N) = -SN2*TI(1)
0197       DB(2,1+2N)  = SGS*TI(1)
0198       DB(2,1+2N)  = CSS*TI(1)
0199       DB(2,1+2N)  = SN2*TI(1)
0200       DB(3,1)          = SN2*TI(1)/2.
0201       DB(3,1+2N)  = SN2*TI(1)/2.
0202       IB(3,1+2N)  = CS2*TI(1)
0203       IF(N.EQ.1) GO TC 161

0204       DB(1,1+I)        = TI(1,1)
0205       DB(2,1+I)        = YG(I,1)
0206       DB(3,1+I)        = -TI(2,1)
0207       DB(1,1+I+2N) = TI(2,1)
0208       DB(2,1+I+2N) = TI(1,1)
0209       DB(3,1+I+2N) = TI(1,1)

0210       DB(1,1+2,I)      = -SG(1,1)
0211       DB(2,1+2,I)      = YG(1,1)
0212       DB(3,1+2,I)      = -TG(2,1)
0213       DB(1,1+3,I+2N)= TG(3,1)
0214       DB(2,1+3,I+2N)= TG(3,1)
0215       DB(3,1+3,I+2N)= TG(4,1)

0216       DB(1,1+I)        = TI(1,1)
0217       DB(2,1+I)        = YG(1,1)
0218       DB(3,1+I)        = -TG(2,1)
0219       DB(1,1+I+2N) = TG(2,1)
```

104

```
0220            DA(3,1)=2.)*2*N)       B.-M43(3))
0221            DM(4)(3.)*(2*N)(1)     ; 1*(3.1(1))

        C
0222       151 CONTINUE
0223            DA(1,N)              C   CSSP*T(N)
0224            DM(1,2*N)                SASP*T(N)
0225            CH(L,3*N)         =  -SN2P*T(N).
0226            DA(2,N)              =  SASP*T(N)
0227            DA(2,2*N)            =  CSSP*T(1)
0228            DA(2,3*N)           =  SN2P*T(1)
0229            DB(3,N)             =  SN2P*T(N)/2.
0230            DB(3,2*N)          = -SN2P*T(N)/2.
0231            CB(3,3*N)           = -CS2P*T(N)
0232            IF(MSINGO)          GO TO 167
0233       161 CONTINUE

        C
        C        UNDERFLOW CHECK  CN  MATRIX
0234            DO 163 K=1,LT3
0235            DO 163 L=1,LT3
0236            IF(DABS(CB(K,L))*LT.1.0D-40)  DB(K,L) = 0.0D 00
0237       163 CONTINUE

        C
        C        INVERT MATRIX  DB
0238            CALL INVT(CB,20,LT3,CFT,1.0D-12,IFANK)

        C
        C        CHECK  FOR  SINGULAR  MATRIX
0239            IF(IRANK.EQ.LT2)  GC  TO 168
0240            MSINGO = .TRUE.
0241            WRITE(6,1470) IFANK,CCT
0242            GO TO 130
0243       167 CONTINUE
0244            WRITE(6,1425) ((DB(K,L),L=1,LT3),K=1,LT3)
0245            MSINGO = .FALSE.
0246            GO TO 199
0247       168 CONTINUE

        C
        C
0248            DC(1)           = -SG1)*TT
0249            DC(2)           = -SG2P*TT
0250            EC(3)           = -SUL2*TT

        C
0251            DO 5 I=1,LAV
0252            SNS     =  SINS(I)
0253            CSS     =  CQSS(I)
0254            SN2     =  SIN2(I)
0255            CS2     =  COS2(I)
0256            DC(1)         DC(1) + SG(1,1)*CSS*T(I) + SG(LAV+1,1)*SNS*T(I)
0257                          -SC(2,LAV+1)*SN2P*T(I)
                  DC(2)       + SG(1,1)-SNS*T(1)  +  SC(LAV+1,1)*CSS*T(I)
                              +SG(2,LAV+1)*SN2P*T(1)
0258        1     DC(3)       + SG(1,1)*T(I)*SN2/2. - SG(LAV+1,1)*SN2*T(I)/2.
0259            DE(3)               SN2P2. + SG(2,LAV+1)*CS2P*T(1)
        C
0260       5 CONTINUE
        C        REDEFINE  COEFFICIENTS  S22  AND  S46
0261            DO 7 K=1,N
0262            T125      = (S(IH+2*N,1)/Y(K))**2
0263            TS225     = (S(I(K*N),1)/SY(K))**2
0264            S125      =  T125 + TS225
                Y(IN)   = (1.0D0+S125*4*((XK-1.)/2.))/S22(K)
```

105

```
0265         S44(K) = ((LADD+S12S0*((XM-1.)/2.))/(4.*(000+C12(K))
0266      7 CONTINUE
           C
0267         DO 8 K=1,LM1
0268         SNS   = SINS(K)
0269         CSS   = CCSS(K)
0270         SN2   = SIN2(K)
0271         CS2   = CCS2(K)
0272         SNSP  = SINS(K+1)
0273         CSSP  = CCSS(K+1)
0274         SN2P  = SIN2(K+1)
0275         CS2P  = CCS2(K+1)
           C
0276         DC(3,3*K-2) =       -(S11(K)*CSS+S22(K)*SN2)*SG(K,1) -
          1   SG(LAY+K,1)*(S12(K)*CSS+S22(K)*SN2) + 2.*S44(K)*SN2P*
          2   SG(K+1,1) + (S12(K+1)*CSSP*S22(K+1)=CSSP*S2(K+1)=SN3P)
          3   4.*SG(K+1,1) + (S12(K+1)*CSSP*S22(K+1)*SN2P)+
          4   SG(LAY+K+1,1)  2.*S44(K+1)*SG12*(LAY+K+1,1)=
          5            SN2P
0277         DC(3,3*K-1) =  -(S11(K)*SNS+S12(K)*CSS)*SG(K,1) - (S12(K)*SNS+
          1   SG(2*LAY+K,1) +
          2   (S11(K+1)*SNSP+S12(K+1)*CSSP)*SG(K+1,1) + (S12(K+1)*SNS+
          3   SNSP*.22(K+1)*CSSP)*SGLAY+K+1,1)+ 2.*S44(K+1)*SNSP)+
          5   *SG12*(LAY+K+1,1)
0278         DC(3,3*K)  =  -(S11(K)-S22(K))*SN2*SG(K,1)/2. -(S12(K)-S22(K))*
          1   SN2*SG(LAY+K,1)/2. - 2.*S44(K)*CS2*SG(2*LAY+K,1) +
          2   (S11(K+1)-S22(K+1))*SN2P*SG(K+1,1)/2. + (S12(K+1)-
          3   S22(K+1))*SN2P*SG((LAY+K+1,1)/2. + 2.*S44(K+1)*CS2P*
          4            SG(2*LAY+K+1,1)
           C
0279       X CONTINUE
           C
0280         DO 9 I=1,LT3
0281         DC(I) =-DC(I)
0282       9 CONTINUE
           C
0283         DO 1 J=1,LT3
0284         DC 1 M=1,LT3
0285       1 DTI(I) = DTI(I) + DC(1,M)*DC(M)
           C
0286         DC 15 I=1,LT3
0287         SG(I,1) = SG(I,1) + DT(I)
0288      15 CONTINUE
           C
0289         C TEST STRESS = 0. IF RELATIVE STRESS / L.E-04
0290         CALL RESTRT(IT,SG,1.0D-06)
0291         IF(IIT.EQ.02) GO TO 145
           C
           C CONVERGENCE CHECK
0291         CALL CONVR(LAY,SG,SG,LM1,CON)
0292     100 CONTINUE
           C
           C STRAIN COMPUTATIONS
0293         DO 540 I = 1,LAY
0294         SNS   = SINS(I)
0295         CSS   = CCSS(I)
0296         SN2   = SIN2(I)
0297         CS2   = CCS2(I)
           C
0298         PII(I) = S11(I)*SG(I,1) + S12(I)*SG(LAY+I,1)
0299     125  (SG12*2EN,1)/TV(IC))=
```

```
0300        TS225 = (SG(I+N,I)/SY(I))**2
0301        S12A = T1(S + TS225
0302        PZ211 = SQ11*SG(I+1) + SG(I+2)*I/I/(Z2(I))*(1.000*SI2)*(1XP-1.)/Z2*))*I(AN
                   -1./Z2(1)
0303        PIZ(I) = SG(I+2**,1)/I2.*G(2(1))*(1.000*SI2)*(1XP-1.)/Z2*))
0304 520    CONTINUE
     C
0305        PP1(I) - PI(I)*CGS + PP2(I)*TGS - PIZ(I)*SP+
0306        PP2(I) = PI(I)*SA + PP2(I)*CGS + PIZ(I)*SN2
0307        PP12(I) = (PP1(I)-PP22(I))*SN2/2.00 + PIZ(I)*CS2
     C
0308 590    CONTINUE
     C
     C
0309        ! APPRATE TESTS
            CALL TAM(ST(I)AV,SG,SGS,REC,RSGM)
     C
     C      OUTPUT IT.
0310 990    CONTINUE
0311        WRITE(6,1325)
0312        WRITE(6,1547)  SG11
0313        WRITE(6,1558)  SG22
0314        WRITE(6,1529)  SG12
0315        WRITE(6,1235)  N1T
0316        WRITE(6,1536)
0317        WRITE(6,1537)
0318        WRITE(6,1538)
0319        DO 120 I = 1,NAV
0320        WRITE(6,1550)  I,SG(I,1),SG(I+N,1),SG((I,*N,1),
                   PP1(I),CP22(I),PP2(I),P1(I),PP2(I),PIZ(I),PIZ(I)
     C
0321 750    CONTINUE
0322        WRITE(6,1589)
     C
0323        ICIASGM-KSL) 790,790,760
     C
     C      MULTIPLICATIVE FACTOR FOR INITIAL ESTIMATE OF SUCEEDING
     C      INCREMENTATION
     C
0324 760    CONTINUE
0325        GO TO I(SG,760), INAT
     C
     C      MATEC OF PREVIOUS SOLUTIONS
0326 762    CONTINUE
0327        DO 765 I=1,ITX
0328        IF(KSC.EQ.1) GO TO 770
0329        IF(ISG(I).EQ.0.0001 GO TO 770
0330        SY(I) = SG(I,1)/SGS(I)
0331        GO TO 765
0332 763    CONTINUE
0333        SY(I) =         SGNT
0334 765    CONTINUE
0335        GO TO 770
     C
0336 768    CONTINUE
0337        DO 766 I=1,ITX
0338        SY(I)*SG(I,1) GO TO 761
0339        ISG(I,1).EQ.0.0001     GO TO 761
0340        SY(I) = SG(I,1)
0341        CO(I) = SG(I,1)*SG(I,1)*SG(I,1)*1537
0342        CO(I)    1.000 + 4SG.*SG(I,1)* SGS(I)/ (I2.I)
0343        GO TO 769
0344 767    CONTINUE
```

FORTRAN IV G LEVEL 21   MAIN   DATE = 74047   16/26/45   PAGE 0004

```
0345          WRITE( 1.0000
0346          746 CONTINUE
          C
0347      C    STOPE STRESS AND STRAIN VALUES
0348          772 CONTINUE
0349              DO 775 J = 1,LAY
0350              SGS(I)    =  SG(I)(  ,I)
0351              SGS(I+ K)  =  SG(I)(  ,I)
0352              SGS(I+2*K) =  SG(I)(CB,I)
0353              PS1(I)  =  P1(I)
0354              PS2(I)  =  P2(I)
0355              P12(I)  =  P12(I)
0356              PS1(I)  -  PS1(I)
0357              PS2(I)  =  PS2(I)
0358              P12(I)  =  P12(I)
              775 CONTINUE
          C
0359      C    INCREMENT APPLIED LOADING
0360              KSG = KSG + 1
0361              SD11 =  SC11 + SPM1
0362              SD22 =  SC22 + SPM2
0363              SD12 =  SC12 + SPM12
0364          GO TO 110
              790 CONTINUE
0365          800 CONTINUE
          C
0366          999 CONTINUE
0367              WRITE(6,1599)
          C
0368         1000 FORMAT (5C15.5)
0369         1010 FORMAT (5I5)
0370         1020 FORMAT (10I5,5I2)
0371         1024 FORMAT (I5,5C15.5)
          C
0372         1420 FORMAT (///  MATRIX IS SINGULAR OF RANK = ',I2,
0373         1425 FORMAT (6C15.5)
          C
0374         1500 FORMAT (* INELASTIC LAMINATE ANALYSIS*////)
0375         1505 FORMAT (1H1,*LAMINATE *,I5)
0376         1510 FORMAT (//// NUMBER OF LAYERS  ',I2)
0377         1513 FORMAT (/ LAYER  1=T4D,2X,V1,11X,SIG X,6FL*,11X,SIG22,11X,
0378                     V12,11X,V21,11X,G12,11X,C12,11X,V1,9X,SIG Y,7)
0379         1515 FORMAT (I4,2X,F13,75,C12,5,4X,7(2X,212.5))
0380         1516 FORMAT (//  EQUATION DEVIATION(//.5)
0381         1517 FORMAT (/  EXPONENT Q = ',C12.5)
0382         1518 FORMAT (/  EXPONENT A = ',C12.5)
0383         1520 FORMAT (//  LOADING EXTERNAL  (APPLIED)
0384         1525 FORMAT (* EXTERNAL APPLIED LOADING(N../)
0385         1527 FORMAT (* F X = ',C12.5)
0386         1528 FORMAT (* F Y = ',C12.5)
0387         1529 FORMAT (* F XY = ',C15.5)
0388         1530 FORMAT (6X2X,*LAMINATE *,F5.2,...)
0389         1537 FORMAT (/ LAYER ...)
0390         1555 FORMAT (I4,1X,2(3I15,5,4.5),1C11.5)
```

108

```
FORTRAN IV G LEVEL  21          MAIN          DATE  23027      16/26/39      PAGE 0010

0191        1561 FORMAT (//51X,'MATERIAL PROPERTIES')
0392        1563 FORMAT ( /34X,'LAYER',34X,'MAT.IX')
0393        1565 FORMAT (/' LAYER',34X,'MAT.IX',1X,'X',1X,'X',1/L
0394        1735 FORMAT (/' SOLUTION FOR STRESS CONVERGES WITHIN',I4,
             1            ' ITERATIONS'/)
0395        1581 FORMAT (' -------------')
0396        1583 FORMAT (( -------------------))
0397        1566 FORMAT (/62X,'----------'/)
0398        1595 FORMAT (1HC)
0399        1599 FORMAT (1H1)
0400             STOP
0401             END
```

109

FORTRAN IV G LEVEL 21   ANGLE   DATE = 12041   16/26/39   PAGE 0001

```
0001        SUBROUTINE ANGLE(IAV,IANG)

C           ASSIGN ANGLES TO VALUES BETWEEN 0 AND 360 P1/4 FLD
C           COMPUTING SIN AND COS

0002        IMPLICIT
0003        DIMENSION
0004        COMMON
0005        DO 12 I = 1,IAV
0006        IANG2 =
0007        IANG =
0008        ANG =
0009        ANG /37.29577951700/
0010        ANGC =
0011        IAVAL =
0012        IF(IAVAL.
0013        IF(IAVAL.
0014        NE
0015        COS
0016        SIN
0017
0018        ...
0019        ...
0020  62    SUMIANG
0021        ISGN
0022        IF(IAVAL.NE.45)GO TO 64
0023        COS
0024        SIN
0025        GO TO 62
0026  64    CONTINUE
0027        IF(IAVAL.LT.45)GO TO 66
0028        IPOA =
0029        IPOA
0030        IPOA =
0031        SIN2(I) =
0032        COS2(I) =
0033        GO TO 69
0034  66    CONTINUE
0035        SIN2(I) =
0036        COS2(I) =
0037  68    CONTINUE
0038        SIN(I) =
0039        COS(I) =
0040  72    CONTINUE
0041        RETURN
0042        END
```

FORTRAN IN C LEVEL 21   CUNVP   DATE = 13027   16/26/14   PAGE 0001

```
0001        SUBROUTINE CUNVP(LAY,S0,S0LPSD,P,Q)

        C
0002        IMPLICIT         REAL*8 (A-H,O-Z)
0003        DIMENSION        SO(20,1),S0LPSD(1),P(20),Q(20)
0004        COMMON           /CCLARN/EPS,JPLC
                             /PRCPADM/IT,NIT
        C
0005        ICON = 1
0006        K = LAY
0007        IJ = LAYER
        C
        C         CONVERGENCE CHECK
0008        GO 375 J3=1,J7
0009        SUM = DABS(S(IJ,J3))-DABS(SO(IJ,J3))
0010        IF(IJ(J3,1).EQ.0.0103) GO TO 330
0011        IF(J3) = DABS(SUM/S(IJ,J3,1))
0012        GO TO 335
0013   330  CONTINUE
0014        DIF(J3) = SUB
0015   335  CONTINUE
0016        IF(DIF(J3).GT.EPS) GO TO 340
0017        GO TO 375
        C
        C         ITERATION CHECK
0018   340  CONTINUE
0019        IF(INT-IJ).NE.0) GO TO 350
0020        ICON = 3
0021        IF(IF(J3).LE.UPHD) GO TO 490
0022        ICON = 4
0023        GO TO 375
        C
        C         DIVERGENCE CHECK
0024   350  IF(DIF(J3).LE.UPHD) GO TO 470
0025        ICON = 4
0026        GO TO 375
        C
        C
0027   370  CONTINUE
0028        ICON = 2
0029   375  CONTINUE
        C
0030        GO TO (500,400,3H2,3HG),ICON
        C
        C         NON-CONVERGENCE DUMP
0031   470  CONTINUE
0032        WRITE(6,1720)
0033        WRITE(6,1722) EPS
0034        GO TO 395
0035   490  CONTINUE
0036        WRITE(6,1730)
0037        WRITE(6,1727) UPHD
0038   495  CONTINUE
0039        NIT = NIT - 1
0040        WRITE(6,1410) NIT,NIT
0041        WRITE(6,1425)
0042        GO 397 I=1,LAY
0043        WRITE(6,1430) J,S(I,1),S(I,2),S(I,3),SO(I,1),SO(I,2),SO(I,3),SO(I,1),P(I),SO(I,N,1),
                             P(I,N),DIF(I),DIF(I,N),DIF(I,2,N)
0044   397  CONTINUE
0045        RETURN
```

111

FORTRAN IV G LEVEL 21   COMP   DATE = 79027   16/26/59

```
0046        400 RETURN )
0047        500 ( - END )

0048       1550 FORMAT (('4,1X,21E(?),24,21,901H,.))
0049       1720 FORMAT (' SOLUTION  + DK  -GXLSS  DOES  NOT CONVERGE.')
0050       1722 FORMAT (' RELATIVE  ERROR   GT., 1., 1.,,)
0051       1733 FORMAT (' IN  STATES  (DIVERGES.)
0052       1741 FORMAT (/1HX,'ITERATION ',13,(,293,9(ITERATION ',13,'))
0053       1776 FORMAT ( ' LEVEL ,4X,'SGR X,SGR Y,MAX(SGR X),Y(MAX(SGM X),SGM X',
                         9X,'SGM Y','HX,'SGR XY','1X,'REF X',4X,'REF Y',HX,
                         'REL(X(/)))
                   ;

0054            C     END
```

112

```
0001        SUBROUTINE LAMISI(LLAY,NS,SC,K,G,KS,M)
       C
0002        IMPLICIT         ...
0003        DIMENSION        ...
0004        ...

            C    TEST 1: STIFFNESS TEST
0005        550 CONTINUE
0006            DO 575 I = 1,LLAY
0007            IF(KSG.EQ.1) GO TO 560
0008            IF(S.LE.EQ.0.0E0) GO TO 560
0009            RATIO = DABS(SM1)/(FP1(I))-FPS1(I)))
0010            IF(RATIO.LT.S1(I)) GO TO 575

            C    TEST 2: TEST ...
0011        560 CONTINUE
0012            ...

0013        575 CONTINUE
            C
0014            WRITE(6,1555)
0015            RETURN
0016        677 IFAIL = 1
0017            KSC = KSGM
0018            GO TO 700
0019        678 IFAIL = 2
0020            KSC = KSGM
0021            GO TO 100
            C
0022        700 CONTINUE
0023            GO TO (1/(QL/03), IFAIL
0024        701 WRITE(6,1450)
0025            GO TO 725
0026        703 WRITE(6,1452)
0027            GO TO 725
0028        725 WRITE(6,1455)
            C
0029       1450 FORMAT(1//, LAMINATE HAS FAILED:  STIFFNESS TEST FAILURE)
0030       1452 FORMAT(1//, LAMINATE HAS FAILED:  SGM X EXCEEDS MAXIMUM)
0031       1455 FORMAT(1//, AT FAILURE)
0032       1555 FORMAT(1//)
            C
0033            RETURN
0034            END
```

113

FORTRAN IV G LEVEL 21   MATRM   DATE = 73027   10/26/79   PAGE 0001

```
0001      SUBROUTINE MATRM(DLAY,SG,F,mat,G1,SY,TV,XN,XM,I)

0002      IMPLICIT
0003      DOUBLE PRECISION  VARS
0004      DIMENSION  G1(20),S1(20),S21(20),S1(20),C22(20),G12(20),
                    S1N2(20),COSS(20),SIN2(20),H13,20),TV(20),SY(20),LL,
                    F(2,20),H13,20),G12(20),TV(20),SY(20)

0005      COMMON /BAENS/S1,S12,C1S2,SINS,COSS

          N = LAY
0006      T12S =  (SG(1)*2*N,1)/TV(1))**2
0007      T12D =  SG(1)*2*N,1))/TV(1) **2
0008      T22S =  (SG(1)*N,1)/TV(1))**2
0009      T120 =  SG(1*N,1)/TV(1) **2
0010      VAL =  SY(1)*1.0D0-60
0011      V2T =  SG(1*2*N,1)*SG(10*N,1)/SV(1)
0012      VAL =  DABS(VAL)
0013      MAT =  LADS(MAT)
0014      IF(RAT.LE.VAL)  RAT = 0.0D-02
0015      T12S =  MAT/SV(1)
0016      C12T =  SG(1*2*N,1)*SG(10*,1)/TV(1))**2
0017      TS225 =  1SG(1*N,1)/SV(1))**2
0018      S12S =  T12S + TS225

          SNS  =  SINS(1)
0020      CSS  =  COSS(1)
0021      SN2  =  SIN2(1)
0022      CS2  =  COS2(1)

0024      F(1,1) = S1(1)*CSS + S21(1)*SNS
0025      F(2,1) = S1(1)*CSS + SNS/E22(1) + SN2/E22(1)*S12S**(XN-1.)/2.)
                 + SN2*(XM-1.)/(2.*E22(1))*S12S**((XM-3.)/2.)*C12S
0026      F(1,3) = -SN2/(2.*G12(1)) - SN2/(2.*G12(1))*S12S**((XM-1.)/2.)
                 - SN2*T12S*(XM-1.)/(2.*G12(1))*S12S**((XM-3.)/2.)
                 + C12S*(XN-1.)/E22(1)

0027      G(1,1) = S1(1)*SNS + S21(1)*CSS
0028      G(2,1) = SNS*S12(1) + CSS/E22(1) + S12S**((XN-1.)/2.)*CSS/E22(1)*T5225
                 + CSS,
                 + (XM-1.)*S12S**((XN-3.)/2.)/E22(1)*T5225

0029      G(3,1) = SN2/(2.*G12(1)) + SN2*S12S**((XN-1.)/2.)/(2.*G12(1))
                 + SN2*(XM-1.)*S12S**((XN-3.)/(2.*G12(1))*T5225
                 + (XN-1.)*S12S/E22(1)

0030      M(1,1) =  S1(1)-S12(1))*SN2/2.
0031      M(2,1) =  S12(1)*SN2/2. - SN2/(2.*E22(1)) - S12S**((XM-1.)/2.)*SN2
                 + /(2.*E22(1))
                 + CS2/(2.*G12(1))

0032      M(3,1) = CS2/(2.*G12(1)) + S12S**((XM-1.)/2.*G12(1))
                 + (XM-1.)*S12S**((XM-3.)/2.)/(2.*G12(1))
                 + (XN-1.)*S12S**((XN-3.)/2.)*SN2/(2.*E22(1))

0033      RETURN
0034      END
```

```
0001          SUBROUTINE RESET(I1,SIGMA,RVAL)

0002          IMPLICIT          REAL*8 (A-H,O-Z)
0003          DIMENSION  SG(20,1)

0004          SUMX = DABS(SG(1,1))
0005          DO 318 K=2,I1
0006          IF(DABS(SG(K,1)).GT.SUMX) SUMX = DABS(SG(K,1))
0007      318 CONTINUE
0008          DO 319 K=1,I1
0009          RAT = DABS(SG(K,1)/SGMX)
0010          IF(RAT.LT.RVAL)  SG(K,1) = 0.0D 00
0011      319 CONTINUE
0012          RETURN
0013          END
```

115

```
0001        SUBROUTINE INPUT2(I1,I1Z,I2Z,V1Z,V2Z,G12,SV,TV,I1)
       C
       C
0002        IMPLICIT REAL*8(A-Z)
0003        DOUBLE PRECISION DSQRT
0004        DIMENSION TI1(20),TC(20),V12(20),V21(20),G1(20),SV(20),
           *  GSA(11),GSA2(11),GAM1(20),TAU(20),TV(20),TY(5)
0005        INTEGER I,J,NUM1,NUM5,I2

0006        READ(5,1002) FF,MUA,GA,VS
0007        READ(5,1002) EM,MUM,GM
0008        READ(5,1005) I2
0009        VM = 1.00 0 - VS
0010        TRM1 = (1.0D0+FF/(1.0D0-VS) * GM
0011        WRITE(6,1956) 1,FF,MUA,GA,VS,MUM,GM,VP
0012        IF(I2.EQ.1) GO TO 30
0013        READ(5,1002) SV(I),TV(I)
0014        G12(I) = TPM1
0015        GO TO 60
0016     30 CONTINUE
0017        READ(5,1002) TY(I)
0018        READ(5,1005) NUM1
0019        READ(5,1002) (TAU(J),J=1,NUM1)
0020        READ(5,1002) (GAM1(J),J=1,NUM1)
0021        READ(5,1002) I,G12(I),SV(I),J=1,I1)
       C
       C        COMPUTE TAUV FROM TABLE
0022        SUMS6 = 0.0D0
0023        SUMS4 = 0.0D0
0024        SUMS3 = 0.0D0
0025        DO 40 J=1,NUM1
0026        SUMS6 = SUMS6 + TAU(J)**2
0027        SUMS4 = SUMS4 + TAU(J)**4
0028        SUMS3 = SUMS3 + GAM1(J)*TAU(J)**2
0029     40 CONTINUE
0030        TAUV = DSQRT(SUMS6/(6*SUMS3-SUMS4))
       C
       C        COMPUTE SV AND G12
0031        GSA2(I) = 0.0D0
0032        SUMS6 = 0.0D0
0033        SUMS4 = 0.0D0
0034        TRM2 = (3.0D0*1.0D0*VS)/(1.0D0*VS*1.0D0*VS)) /2.0D0
0035        DO 50 J=1,I1
0036        GSA(J) = TRM1/((SG12(J)/TAUV)**2+2.0D0)
0037        SUMS6 = SUMS6 + SG12(J)**6
0038        SUMS4 = SUMS4 + SG12(J)*(GSA(J)*TAU(J)+1.0D0)
0039     50 CONTINUE
0040        SV(I) = DSQRT(SUMS6/SUMS4)
0041        G12(I) = GSA(I)
       C
       C
0042     40 CONTINUE
0043        PI = 3.1415900
0044        KF = FF/12.0D0*(1.0D0*MU*(1.0D0-2.0D0*VS)))
0045        KM = FM/12.0D0*(1.0D0*MUM*(1.0D0-2.0D0*VM))
0046        M = (KF*(PI*4.0D0*MUM) + (-000*K*VP*(1-4.0D0*VF)*4MM)/
           *    (KF*(PI*4.0D0*MUM) + 2.0D0*K*4*(VF*VP*(7.0D0*4MF)))
0047        KK = KM/2.0D0*(1.1(3.0D0*(1/4.0D0 + ...
```

```
               ... = 3.000*M/ZIP*(3.000C-P1)*M1)
      0048          E11(I)) = FF*V1 + FM*(1.000-V1)
      0049          V12(I) = FU*FY1 + FM*(1.000-V1)
      0050          M1/3  = FU*FY1 + MU*FV1*(1.-MU*-V1>((I*(FM/F)I(2)))
      0051          E22(I) = 2.000 O*KK*M*11J1*(1.000-MU23)/(11(1))/4.000*KK*
                             V12(I)=-2)
      0052          V21(I) = V12(I)*E22(I)/E11(I))
      0053    1002 FORMAT (5F15.5)
      00..         1002 FORMAT (5I5)
      0055    1567 FORMAT (I4,2(7X, 4(D11.4/2X)))
               C
      0056         RETURN
      0057         END
```

```
0001       SUBROUTINE PXRULE(A,C,N,NHA,N(CLA,NCQ(A,RCQR(N,CIR-(N/,NC(LD)    F0600001

       C
       C

0002       DIMENSION Z(NCQR,NC(H2),P(PC(NA),NCQ(LA,RCQR(S,CIR-(N/,NC(LD)    F0600002
0003       DOUBLE PRECISION Z,X,P,F                                        F0600003
0004       DO 20 I=1,N2                                                    F0600004
0006       DO 10 J=1,NP                                                    F0600005
0007       X=0.                                                           F0600006
0027    10 Z(I,J)=X,AB                                                     F0600007
0008       K=X+A(I,K)*H+H+H+H+(J)                                         F0600008
           X(J,J)=X                                                       F0600009
0100    10 Z(I,J)=X                                                       F0600010
0100    20 CONTINUE                                                       F0600011
0110       RETURN
1100       END
```

```
0001          SUBROUTINE INVERT(A,ROW,...)
0002          DOUBLE PRECISION PIV,...
0003          DIMENSION IROW(150),...
0004          PIV=1.
0005          SUM=0.
0006          DO 10 I = 1,A
0007          DO 10 J = 1,A
0008   10     SUM=SUM+A(I,J)**2
0009          SUP=DSQRT(SUM)
0010          DMA = N+2
0011          RMS=SUM/DMA
0012          TOL=EPS*RMS
0013          DO 20 I = 1,A
0014          IW(I)=0
0015   20     IC(I)=0
0016          S=0
0017          M = N
0018   30     I=0
0019          J=0
0020          TEST=0.0
0021          DO 50 K = 1,N
0022          IF(IW(K).NE.0)GO TO 50
0023          DO 40 L = 1,M
0024          IF(IC(L).NE.0)GO TO 40
0025          ...
0026          I=K
0027          J=L
0028          TEST=K
0029   40     CONTINUE
0030   50     CONTINUE
0031          PIV=A(I,J)
0032          BETA=... DETA = 0.30 00
0033          IF(...) GO TO 130
0034          PIV = 1.0D0/PIV
0035          A(I,J)=PIV
0036          DO 60 K = 1,N
0037          IF(K.EQ.I) GO TO 60
0038   60     CONTINUE
0039          DO 70 K = 1,A
0040   70     CONTINUE
0041   80     CONTINUE
0042   100    CONTINUE
0043   110    ...
0044          ...
0045          ...
0046          ...
```

```
0059         TEMP=SF(K,L)
0060         A(P,L)=A(K,L)                         PL200059
0061    126  A(K,L)=TEMP                           R120060
0062         DO 140 L = 1,A                        PL200061
0063         TEMP=A(L,M)                           PL200062
0064         A(L,M)=A(L,1)                         PL200063
0065    130  A(L,1)=TEMP                           PL200064
0066         TC(2)=R                               PL200065
0067         JC(K)=P                               PL200066
0068    140  CONTINUE                              PL200067
0069    150  CRATIKES                              PL200068
0070         RETURN                                PL200069
0071         END                                   PL200070
```

SAMPLE PROBLEM

LAMINATE 1

NUMBER OF LAYERS = 3

LAYER	THETA	T	L11	L22	V12	V21	G12	STG Y	TAU Y
1	-45	0.333330-00	0.240220 08	0.240220 07	0.255000 00	0.193510-01	0.987000 06	0.100000 05	0.500000 04
2	0	0.255000-00	0.287550 08	0.287550 07	0.255000 00	0.153810-01	0.987000 06	0.100000 05	0.500000 04
3	45	0.333330 00	0.287550 08	0.287550 07	0.255000 01	0.153810-01	0.987000 06	0.100000 05	0.500000 04

EQUATION PARAMETERS

EXPONENT M = 0.500000 01
EXPONENT N = 0.500000 01

LOADING INCREMENTATION

EXTERNAL APPLIED LOADING

F X = 0.500000 00
F Y = 0.000000 00
F XY = 0.500000 04

SOLUTION FOR STRESS CONVERGES WITHIN 2 ITERATIONS

LAYER	STRESS SUM X	SUM Y	SUM XY	STRAIN (LAMINATE AXES) EPS X	EPS Y	EPS XY	STRAIN (LAYER AXES) EPS X	EPS Y	EPS XY
1	-0.116650 03	0.400070 03	0.249300 03	0.616730-03	-0.302590-03	0.470920-03	-0.414630-03	0.326990-03	0.358860-03
2	0.181190 03	-0.618000 03	0.094960 03	0.414730-03	-0.102590-03	0.470920-03	0.414730-03	-0.302590-03	0.470920-03
3	0.150000 03	-0.002210 03	-0.042260 03	0.414730-03	-0.302590-03	0.470920-03	0.326990-03	-0.414630-03	-0.358660-03

EXTERNAL APPLIED LOADING

F X = 0.100000 05
F Y = 0.000000 00
F XY = 0.100000 05

SOLUTION FOR STRESS CONVERGES WITHIN 2 ITERATIONS

LAYER	STRESS SUM X	SUM Y	SUM XY	STRAIN (LAMINATE AXES) EPS X	EPS Y	EPS XY	STRAIN (LAYER AXES) EPS X	EPS Y	EPS XY

EXTERNAL APPLIED LOADING

F X =   0.150000D 05
F Y =   0.000000D 00
F XY = 0.150000D 05

SOLUTION FOR STRESS CONVERGES WITHIN 2 ITERATIONS

		STRESS			STRAIN (LAMINATE AXES)			STRAIN (LAYER AXES)		
LAYER	SGM X	SGM Y	SGM XY		EPS X	EPS Y	EPS XY		EPS X	EPS Y	EPS XY
1	-0.381640 05	0.236990 04	0.153490 04		0.127480-02	-0.904720-03	0.143330-02		-0.127830-02	0.150840-02	0.111980-02
2	0.303600 05	-0.114470 04	0.020777 04		0.127440-02	-0.904740-03	0.143340-02		0.127480-02	-0.904720-03	0.143340-02
3	0.492520 05	-0.163890 04	-0.160900 04		0.127480-02	-0.904720-03	0.143330-02		0.198860-02	-0.127860-02	-0.111980-02

EXTERNAL APPLIED LOADING

F X =   0.000000D 05
F Y =   0.000000D 00
F XY = 0.000000D 05

SOLUTION FOR STRESS CONVERGES WITHIN 2 ITERATIONS

		STRESS			STRAIN (LAMINATE AXES)			STRAIN (LAYER AXES)		
LAYER	SGM X	SGM Y	SGM XY		EPS X	EPS Y	EPS XY		EPS X	EPS Y	EPS XY
1	-0.404030 05	0.268590 04	0.222250 04		0.142240-02	-0.132600-02	0.192470-02		-0.172800-02	0.212140-02	0.182550-02
2	0.034936 05	-0.145490 04	0.263190 04		0.172270-02	-0.132480-02	0.192470-02		0.172290-02	-0.132890-02	-0.152470-02
3	0.036710 05	-0.235490 04	-0.267510 04		0.172260-02	-0.132850-02	0.192780-02		0.212000-02	-0.172900-02	-0.152550-02

EXTERNAL APPLIED LOADING

F X =   0.000000 05
F Y =   0.000000 00
F XY = 0.000000 05

SOLUTION FOR STRESS CONVERGES WITHIN 2 ITERATIONS

STRESS   STRAIN (LAMINATE AXES)   STRAIN (LAYER AXES)

LAYER	SGM X	SGM Y	SGM XY	EPS X	EPS Y	EPS XY	EPS X	EPS Y	EPS XY
1	-0.262420 05	0.272700 04	0.272180 04	0.181210-02	-0.172610-02	0.262670-02	-0.218610-02	0.225540-02	0.194470-02
2	0.301960 03	-0.301801D 04	0.302728D 04	0.211800-02	-0.171070-02	0.242060-02	0.217860-02	0.171070-02	-0.242060-02
3	0.157190 05	-0.246490D 04	-0.277607E 04	0.211850-02	-0.171270-02	0.242760-02	0.205760-02	-0.210600-02	-0.194460-02

EXTERNAL APPLIED LOADING

F X = 0.35000D 05
F Y = 0.00000D 00
F XY = 0.35000D 05

SOLUTION FOR STRESS CONVERGES WITHIN 2 ITERATIONS

STRESS   STRAIN (LAMINATE AXES)   STRAIN (LAYER AXES)

LAYER	SGM X	SGM Y	SGM XY	EPS X	EPS Y	EPS XY	EPS X	EPS Y	EPS XY
1	-0.753481 05	0.303470 04	0.303410D 04	0.304740-02	-0.261670D-02	0.292600-02	-0.362760D-02	0.318740-02	0.237470-02
2	0.734710 05	-0.199270D 04	0.361900D 04	0.260420-02	0.210700D-02	0.294010D-02	0.260230-02	-0.210710-02	0.294010-02
3	0.309600 05	-0.273060D 04	-0.311430D 04	0.260220-02	-0.211070D-02	0.294020-02	0.318780-02	-0.205370-02	-0.237460-02

EXTERNAL APPLIED LOADING

F X = 0.35000D 05
F Y = 0.00000D 00
F XY = 0.35000D 05

SOLUTION FOR STRESS CONVERGES WITHIN 2 ITERATIONS

STRESS   STRAIN (LAMINATE AXES)   STRAIN (LAYER AXES)

LAYER	SGM X	SGM Y	SGM XY	EPS X	EPS Y	EPS XY	EPS X	EPS Y	EPS XY
1	-0.888660 05	0.353470D 04	0.353600D 04	0.311610-02	-0.351620D-02	0.362260-02	-0.313410-02	0.372010D-02	0.281350-02
2	0.896590 05	-0.222100D 04	0.349480D 04	0.311000-02	0.273520D-02	0.342260-02	0.311180-02	-0.215720-02	0.182220-02
3	0.310720 05	-0.301260D 04	-0.323060D 04	0.311110-02	-0.273270D-02	0.342240-02	0.332060-02	-0.311420-02	-0.281360-02

EXTERNAL APPLIED LOADING

123

F X = 0.000000 00
F Y = 0.000000 00
F XY = 0.000000 00

STRESS

LAYER	SGM X	SGM Y	SGM XY
1	-0.102440 04	-0.424640 04	0.271710 04
2	0.107470 04	-0.245960 04	0.243031 04
3	0.121460 04	-0.324760 04	-0.303180 04

SOLUTION FOR STRESS CONVERGES WITHIN 2 ITERATIONS

STRAIN
(LAMINATE AXES)

	EPS X	EPS Y	EPS XY
	0.365390-02	-0.293300-02	0.372060-02
	0.350330-02	-0.294300-02	0.342610-02
	0.323570-02	-0.243230-02	0.342601-02

STRAIN
(LAYER AXES)

	EPS X	EPS Y	EPS XY
	0.360010-02	0.042320-02	0.373950-02
	0.353590-02	-0.071330-02	0.352610-02
	0.442240-02	-0.100030-02	0.325930-02

EXTERNAL APPLIED LOADING

F X = 0.495000 05
F Y = 0.000000 00
F XY = 0.495000 05

SOLUTION FOR STRESS CONVERGES WITHIN 2 ITERATIONS

STRESS

LAYER	SGM X	SGM Y	SGM XY
1	-0.116100 06	0.100000 04	0.390900 04
2	0.116160 06	-0.205630 04	0.456030 04
3	0.186630 06	-0.149670 04	-0.449560 04

STRAIN
(LAMINATE AXES)

	EPS X	EPS Y	EPS XY
	0.406390-02	-0.315840-02	0.449240-02
	0.406300-02	-0.233640-02	0.449320-02
	0.406360-02	-0.335890-02	0.449240-02

STRAIN
(LAYER AXES)

	EPS X	EPS Y	EPS XY
	-0.407490-02	0.478490-02	0.371140-02
	0.406390-02	-0.333840-02	0.443210-02
	0.478480-02	-0.443030-02	-0.371120-02

EXTERNAL APPLIED LOADING

F X = 0.395000 05
F Y = 0.000000 00
F XY = 0.500000 05

SOLUTION FOR STRESS CONVERGES WITHIN 2 ITERATIONS

STRESS

LAYER	SGM X	SGM Y	SGM XY
1	0.126100 06	0.000000 04	0.000000 05
2	0.126960 06	-0.624990 04	0.495060 04
3	0.151920 06	-0.371600 04	-0.433390 04

STRAIN
(LAMINATE AXES)

	EPS X	EPS Y	EPS XY
	0.405360-02	-0.375160-02	0.495960-02
	0.455530-02	-0.379160-02	0.495360-02
	0.455640-02	-0.371960-02	0.495990-02

STRAIN
(LAYER AXES)

	EPS X	EPS Y	EPS XY
	0.479090-02	0.373740-02	0.416830-02
	0.455240-02	-0.479640-02	0.495930-02
	0.331160-02	-0.463290-02	-0.416810-02

EXTERNAL APPLIED LOADING

F X = 0.500000E 05
F Y = 0.000000E 00
F XY = 0.500000E 05

SOLUTION FOR STRESS CONVERGES WITHIN 2 ITERATIONS

		STRESS			STRAIN (LAMINATE AXIS)			STRAIN (LAYER AXIS)		
LAYER	SGM X	SGM Y	SGM XY	EPS X	EPS Y	EPS XY	EPS X	EPS Y	EPS XY

EXTERNAL APPLIED LOADING

F X = 0.000000E 00
F Y = 0.000000E 00
F XY = 0.500000E 05

SOLUTION FOR STRESS CONVERGES WITHIN 2 ITERATIONS

		STRESS			STRAIN (LAMINATE AXIS)			STRAIN (LAYER AXIS)		
LAYER	SGM X	SGM Y	SGM XY	EPS X	EPS Y	EPS XY	EPS X	EPS Y	EPS XY

EXTERNAL APPLIED LOADING

F X = 0.000000E 00
F Y = 0.000000E 00
F XY = 0.000000E 05

SOLUTION FOR STRESS CONVERGES WITHIN 2 ITERATIONS

STRESS   STRAIN (LAMINATE AXIS)   STRAIN (LAYER AXIS)

LAYER   SIG X   SIG Y   SIG XY   EPS X   EPS Y   EPS XY   EPS X   EPS Y   EPS XY

1   -0.171531 06   0.072520 06   0.464949 06   0.000300-02 -0.031610-02   0.069700-02 -0.602370-02   0.070050-02   0.070070-02
2   0.171770 06 -0.038250 06   0.034480 05   0.000370-02 -0.031610-02   0.040080-02   0.000370-02 -0.031610-02   0.040070-02
3   0.019720 06 -0.024870 06 -0.050103 06   0.000340-02 -0.051210-02   0.040730 02   0.040040-02   0.060239-02 -0.056220-02

LAMINATE HAS FAILED, SIG X EXCEEDS MAXIMUM

AT FAILURE
EXTERNAL APPLIED LOADING

F X =   0.700000 05
F Y =   0.000000 00
F XY =   0.000000 00

SOLUTION FOR STRESS COMPONENTS WITHIN THE LAMINATES

STRESS   STRAIN
                        (LAMINATE AXES)   (LAMINATE AXES)

LAYER   SIG X   SIG Y   SIG XY   EPS X   EPS Y   EPS XY   EPS X   EPS Y   EPS XY

1   -0.170000 05   0.000000 00   0.000000 00   0.000100-02   0.072000-02   0.070110-02 -0.000170-02   0.000020-02 -0.003820-02
2   0.180000 05 -0.030000 05 -0.000260 05   0.030030-02 -0.051200-02   0.009770-02   0.009000-02 -0.035270-02   0.000170-02
3   0.030000 05 -0.021800 05   0.000200-02 -0.055710-02   0.000111-02   0.048000-02   0.000590-02 -0.000390-02   0.003310-02

"The aeronautical and space activities of the United States shall be conducted so as to contribute ... to the expansion of human knowledge of phenomena in the atmosphere and space. The Administration shall provide for the widest practicable and appropriate dissemination of information concerning its activities and the results thereof."

—NATIONAL AERONAUTICS AND SPACE ACT OF 1958

NASA SCIENTIFIC AND TECHNICAL PUBLICATIONS

TECHNICAL REPORTS: Scientific and technical information considered important, complete, and a lasting contribution to existing knowledge.

TECHNICAL NOTES: Information less broad in scope but nevertheless of importance as a contribution to existing knowledge.

TECHNICAL MEMORANDUMS: Information receiving limited distribution because of preliminary data, security classification, or other reasons. Also includes conference proceedings with either limited or unlimited distribution.

CONTRACTOR REPORTS: Scientific and technical information generated under a NASA contract or grant and considered an important contribution to existing knowledge.

TECHNICAL TRANSLATIONS: Information published in a foreign language considered to merit NASA distribution in English.

SPECIAL PUBLICATIONS: Information derived from or of value to NASA activities. Publications include final reports of major projects, monographs, data compilations, handbooks, sourcebooks, and special bibliographies.

TECHNOLOGY UTILIZATION PUBLICATIONS: Information on technology used by NASA that may be of particular interest in commercial and other non-aerospace applications. Publications include Tech Briefs, Technology Utilization Reports and Technology Surveys.

Details on the availability of these publications may be obtained from:

SCIENTIFIC AND TECHNICAL INFORMATION OFFICE

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
Washington, D.C. 20546

END

FILMED

DTIC